MAE384, Spring 2020 Homework #3

The deadline of this assignment will be announced by instructor. Please follow the rules on collaboration as given in Homework #1. A statement on collaboration is required for all reports. Please include computer code(s) in the report.

In this assignment, we use "h" (as adopted in the textbook) instead of "Δx" (as used in the lectures) to denote the grid spacing for a finite difference method. In all problems, the grid is assumed to be uniform (i.e., h is a constant).

Prob 1. (4 points)
In this problem, different numerical methods will be used to evaluate the first derivative of the function (which we encountered in HW1-Prob1),

$$f(x) = e^{[\sin(e^x)]} + 0.1x^2 - 2\,,$$

over the interval of $2 \leq x \leq 4$. First, find $f'(x)$ analytically as a preparation for the following tasks.

(a) Choose the interval of discretization as $h = 0.01$ and evaluate $f'(x)$ numerically at $x = 2.0, 2.01, 2.02, 2.03, ..., 3.98, 3.99$, and 4.0. First, perform the calculation using the 2-point central difference scheme (3rd formula in Table 8-1 in p. 318). Then, repeat the calculation using the 4-point central difference scheme (4th formula in Table 8-1 in p. 318). Plot the numerically evaluated and analytically derived $f'(x)$ over the interval of $2 \leq x \leq 4$. Please collect all three curves (or sets of symbols) - one analytic and two numerical - in a single plot. It is recommended that the analytic formula be plotted as connected line, and numerically obtained derivatives plotted as symbols. See Matlab Example 41-44 for useful commands for making a clear plot.

(b) Repeat (a) but now choose $h = 0.005$ and evaluate $f'(x)$ at $x = 2.0, 2.005, 2.01, 2.015, 2.02, ..., 3.985, 3.99, 3.995$, and 4.0. Again, the computation will use the 2nd-order central and 4th order central finite difference schemes, with the derivatives obtained by the numerical methods plotted against the analytically derived $f'(x)$.

(c) Define the numerical error at a given location of x as <u>numerically evaluated $f'(x)$ minus the exact (analytically derived) value of $f'(x)$</u>. (Note: No need to take the absolute value of this difference. The error defined here can be positive or negative.) Make two plots:

> (i) Given the results from Part (a), plot the numerical error as a function of x for the two finite different methods. Collect the two curves in the same plot.
>
> (ii) Repeat (i) but for the results from Part (b) with h refined to 0.005.

Briefly discuss how the magnitude of numerical error changes when h is refined from 0.01 to 0.005. Does the change in error with respect to the refinement of h reflect the expectation that the two finite difference schemes are "second-order" and "fourth-order", respectively?

[Note: All plots for this problem must be made over the interval of $2 \leq x \leq 4$, including the end points, and must be properly labeled. A deduction will be assessed on plots that are of poor quality.]

Prob 2. (2 points)
(a) All of the formula in Table 8-1 have a truncation error of $O(h)$, $O(h^2)$, or $O(h^4)$. Try to derive a four-point, non-centered, finite difference formula for the <u>first derivative</u> of $f(x)$ at $x = x_i$ that has a truncation error of $O(h^3)$. Moreover, the formula must have the following form:

$$f'(x_i) = \frac{Af(x_{i-1}) + Bf(x_i) + Cf(x_{i+1}) + Df(x_{i+2})}{h} + O(h^3)\,.$$

In other words, the derivative of $f(x)$ at $x = x_i$ must be represented by the combination of the values of $f(x)$ at x_i itself, two neighboring grid points to its right, and one grid point to its left, as illustrated in Fig. 1. <u>Show your procedure</u>. Clearly state what the values of A, B, C, and D are.

(b) After obtaining the finite difference formula in Part (a), use it to repeat the calculation in Prob 1(b) (i.e., setting $h = 0.005$ to evaluate the derivative of $f(x)$ in Prob 1 over the interval of $2 \le x \le 4$.). Plot the numerical error as a function of x in the same fashion as in Prob 1(c). Given that the numerical scheme is 3rd-order, we expect it to be superior to the 2nd-order scheme but inferior to the 4th order scheme used in Prob 1. Is this true based on your calculations of numerical errors? To answer this question, you may superimpose the numerical errors for the 3rd-order scheme to the plot for Prob 1(c)-Part (ii).

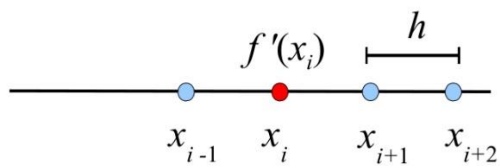

Fig. 1

Prob 3 (2 points)
Derive a five-point, non-centered, finite difference formula for the <u>second derivative</u> of $f(x)$ at $x = x_i$ that has a truncation error of $O(h^3)$. Moreover, the formula must have the following form:

$$f''(x_i) = \frac{Af(x_{i-3}) + Bf(x_{i-2}) + Cf(x_{i-1}) + Df(x_i) + Ef(x_{i+1})}{h^2} + O(h^3)\,.$$

In other words, the second derivative of $f(x)$ at $x = x_i$ must be represented by the combination of the values of $f(x)$ at x_i itself, one grid point to its right, and the neighboring three grid points to its left, as illustrated in Fig. 2. <u>Show your procedure</u>. Clearly state the values of A, B, C, D, and E in the final answer.

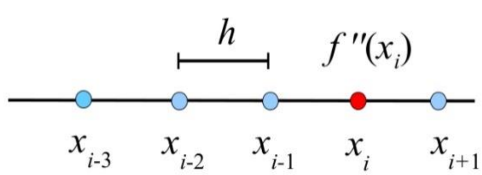

Fig. 2