MAE384, Spring 2022 Homework #4

A statement of collaboration is required. If there is no collaboration, write "No collaboration".
In this assignment, we use "*h*" (as adopted in the textbook) instead of "Δx" (as used in the lectures) to denote the grid spacing for a finite difference method. In all problems, the grid is assumed to be uniform (i.e., *h* is a constant).

Problem 1 (2 points)
All of the formulas in Table 8-1 in the textbook have a truncation error of $O(h)$, $O(h^2)$, or $O(h^4)$. Try to derive a four-point, non-centered, finite difference formula for the first derivative of $f(x)$ at $x = x_i$ that has a truncation error of $O(h^3)$. Moreover, the formula must have the following form:

$$f'(x_i) = \frac{Af(x_{i-2}) + Bf(x_{i-1}) + Cf(x_i) + Df(x_{i+1})}{h} + O(h^3) \, .$$

In other words, the derivative of $f(x)$ at $x = x_i$ must be represented by the combination of the values of $f(x)$ at x_i itself, one neighboring grid point to its right, and two grid points to its left, as illustrated in Fig. 1. Show your procedure. Please clearly state the values of *A*, *B*, *C*, and *D*.

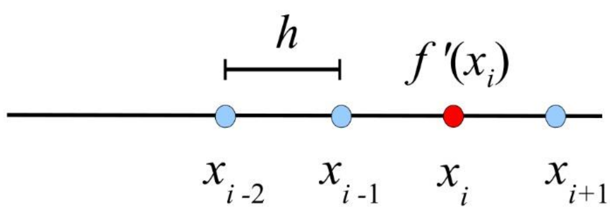

Fig. 1

Problem 2 (5 points)
In this problem, different numerical methods will be used to evaluate the first derivative of the function,

$$f(x) = e^{[\cos(e^x)]} + 0.1x^2 \, ,$$

over the interval of $2 \le x \le 4$. First, find $f'(x)$ analytically as a preparation for the following tasks.

(a) Choose the grid spacing of discretization as $h = 0.01$ and evaluate $f'(x)$ numerically at $x = 2.0$, 2.01, 2.02, 2.03, ..., 3.98, 3.99, and 4.0. Perform the numerical differentiation using the following three finite-difference schemes:

(i) 2nd-order central finite-difference scheme (3rd formula in Table 8-1 in p. 318 of textbook).
(ii) 3rd-order non-central finite-difference scheme as obtained from Problem 1.
(iii) 4th-order central finite-difference scheme (4th formula in Table 8-1 in p. 318).

Plot the numerically evaluated and analytically derived $f'(x)$ over the interval of $2 \le x \le 4$. Please collect all four curves - one analytic and three numerical - in a single plot. It is recommended that the analytic formula be plotted as connected line, and numerically obtained derivatives plotted as symbols. See Matlab Example 41-44 in Beginner's Guide for useful commands for making a clear plot.

(b) Repeat (a) but now choose $h = 0.005$ and evaluate $f'(x)$ at $x = 2.0, 2.005, 2.01, 2.015, 2.02, ..., 3.985, 3.99, 3.995$, and 4.0. Again, the computation will use the 2nd-order, 3rd-order, and 4th-order finite difference schemes, with the derivatives obtained by the numerical methods plotted against the analytically derived $f'(x)$.

(c) Define the numerical error at a given location of x as <u>numerically evaluated $f'(x)$ minus the exact (analytically derived) value of $f'(x)$</u>. (Note: No need to take the absolute value of this difference. The error defined here can be positive or negative.) Make two plots:

> (i) Given the results from Part (a), plot the numerical error as a function of x for the three finite different methods. Collect the three curves in the same plot and label them clearly.
>
> (ii) Repeat (i) but for the results from Part (b) with h refined to 0.005.

Briefly discuss how the magnitude of numerical error changes when h is refined from 0.01 to 0.005. Does the change in error with respect to the refinement of h reflect the expectation that the finite difference schemes are "2nd-order", "3rd-order", and "4th-order"?

[Note: All plots for this problem must be made over the interval of $2 \le x \le 4$, including the end points, and must be properly labeled. A deduction will be assessed on plots that are of poor quality.]

Problem 3 (3 points)
Derive a five-point, non-centered, finite difference formula for the <u>second derivative</u> of $f(x)$ at $x = x_i$ that has a truncation error of $O(h^3)$. Moreover, the formula must have the following form:

$$f''(x_i) = \frac{Af(x_{i-1}) + Bf(x_i) + Cf(x_{i+1}) + Df(x_{i+2}) + Ef(x_{i+3})}{h^2} + O(h^3)\,.$$

In other words, the second derivative of $f(x)$ at $x = x_i$ must be represented by the combination of the values of $f(x)$ at x_i itself, one grid point to its left, and the neighboring three grid points to its right, as illustrated in Fig. 2. <u>Show your procedure</u>. Please clearly state the values of A, B, C, D, and E.

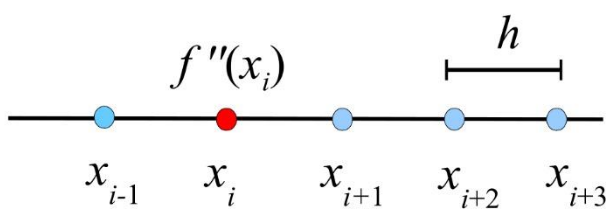

Fig. 2