MAE384, Spring 2022 Homework #6

A statement of collaboration is required. Uses of Matlab built-in functions for solving ODEs, such as **ode23** and **ode45**, are NOT allowed for finding the numerical solutions in this homework. Please include your own code in the report. There is no restriction on the approaches and tools to use to find the *analytic* solutions.

Problem 1 (6 points)
For $u(x)$ defined on $x \geq 0$, consider the initial value problem,

$$\frac{du}{dx} = -x^3 u + x\, e^{-\frac{x^4}{4}} \ , \qquad u(0) = 1 \ ,$$

(a) Find the analytic solution, which will be used to validate the numerical solutions. (You may use any methods or tools to obtain the analytic solution, but do describe how you obtain it.)
(b) Solve the initial value problem using the following three methods ("Δx" is "h" in the textbook):

(I) Euler's implicit method, with $\Delta x = 0.1$
(II) Modified Euler method, with $\Delta x = 0.1$
(III) Third-order Runge-Kutta method, with $\Delta x = 0.1$

In each case, find the numerical solution over the interval of $0 \leq x \leq 2.5$. Plot the analytic solution and the numerical solutions using methods (I), (II), and (III) over the interval of $0 \leq x \leq 2.5$. Please collect all four curves in one plot and clearly label the curves.
(c) From the result of **(b)**, compute the error of the numerical solution (as a function of x), defined by

$$E(x) = u_N(x) - u_E(x) \, ,$$

where u_N and u_E are the numerical solution and analytic solution. Plot $E(x)$ over the interval of $0 \leq x \leq 2.5$ for the three methods used in **(b)**. Please collect the three curves in one plot.

Problem 2 (4 points)
Consider the following initial value problem for $u(x)$ defined on $x \geq 0$,

$$u''' + u'' - 4u' - 4u = 0, \quad \text{("prime" is the derivative, d/d}x\text{)}$$

$$u(0) = 4 \ , \ u'(0) = -2 \ , \ u''(0) = 10 \ .$$

(a) Find the analytic solution, which will be used to validate the numerical solutions. (You may use any methods or tools to obtain the analytic solution, but do describe how you obtain it.)
(b) Solve the initial value problem by first converting the original system into a system of first-order ODEs, then solving the latter using the following two methods ("Δx" is "h" in the textbook):

(I) Euler's explicit method, with $\Delta x = 0.03$
(II) Mid-point method, with $\Delta x = 0.03$

In each case, find the numerical solution over the interval of $0 \leq x \leq 0.6$. Plot the analytic solution and the numerical solutions using methods (I) and (II) over the interval of $0 \leq x \leq 0.6$. Collect all three curves in one plot and clearly label the curves.

Note: For both problems in this assignment, the numerical solutions are the main deliverables. An incorrect or missing analytic solution will be assessed a relatively minor deduction (~ 0.5 point).