MAE 384 Homework 6

This problem set has 2 pages plus a list of the useful finite difference schemes in page 3.

1. Solve the initial value problem,

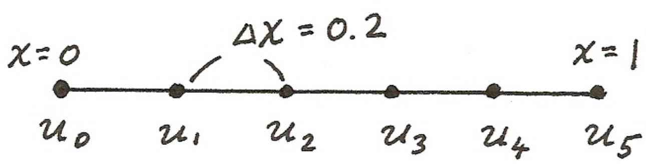
$$\frac{du}{dx} = -u^2 + \sin(x) \quad , u(0) = 0 \ ,$$

to obtain $u(0.4)$ by using **(a)** Euler's explicit method with $\Delta x = 0.1$, **(b)** Classical 4th order Runge-Kutta method with $\Delta x = 0.2$. **2 points**

2. Given the boundary value problem

$$\frac{d^2u}{dx^2} - 3\frac{du}{dx} + 2u = 0 \quad , \ u(0) = 0 \ , u(1) = 1 - e \quad (e = 2.7182818...) \ ,$$

(a) Discretize the equation with $\Delta x = 0.2$ (see illustration below for the grid) and derive a finite difference scheme for the system, using *three-point central difference* scheme for d^2u/dx^2 and *two-point central difference* scheme for du/dx. The resulted system should have the form, [**A**] [**u**] = [**b**], where [**A**] is a matrix and [**u**] and [**b**] are vectors, with [**u**] containing the values of u at appropriate grid points.
(b) Solve the finite difference system in (a) to obtain the numerical solution. Also, solve the original equation analytically. Compare the numerical and exact solutions by plotting them together. (We would not expect an excellent agreement between the two, since Δx = 0.2 is a rather coarse resolution for this problem.)
(The solution for the matrix problem in (b) can be assisted by Matlab. You are however encouraged to solve it by hand; It might help your preparation for the final exam.)
4 points



$u_i \equiv u(i\Delta x)$

3. Find the general solution of the equation

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0 \ ,$$

using the method of *separation of variables*. **2 points**

4. The following equation for $u(x, t)$,

$$\frac{\partial u}{\partial t} = A\frac{\partial^3 u}{\partial x^3} - 0.1u$$

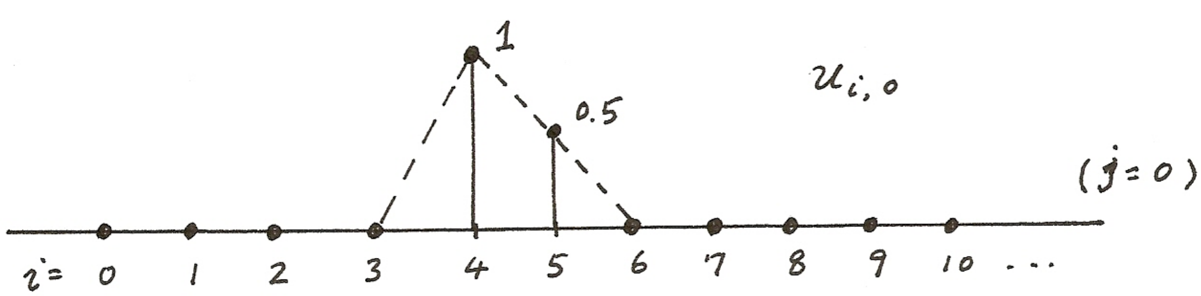
is defined on $x \in (-\infty , \infty)$, $t \in [0 , \infty)$, and with A as a constant.
(a) Derive a finite difference scheme for the equation using (i) *two-point forward difference* scheme for $\partial u/\partial t$, (ii) *four-point central difference* scheme for $\partial^3 u/\partial x^3$, and using Δx and Δt (both constant) as the grid spacing in x and t, respectively. The resulted equation can be written as

$$u_{i,j+1} = P\, u_{i-2,j} + Q\, u_{i-1,j} + R\, u_{i,j} + S\, u_{i+1,j} + T\, u_{i+2,j} \;,$$

where $u_{i,j} \equiv u(i\Delta x, j\Delta t)$. What are the P, Q, R, S, and T in the above equation, as a function of (A, Δx, Δt)?
(b) Given $A = 0.8$, $\Delta x = 0.2$, $\Delta t = 0.1$, and the initial condition at $t = 0$ ($j = 0$) as (see illustration below)

$$u_{4,0} = 1 \;, u_{5,0} = 0.5, \text{ and } u_{i,0} = 0 \text{ for all other } i \;,$$

use your finite difference scheme in (a) to integrate the equation forward for 2 steps to obtain the numerical solution for $u(x, t{=}0.2)$, i.e., the values of $u_{i,j}$ at $j = 2$. Sketch the solution at $t = 0.2$ in the same fashion as the initial state illustrated below.
(Note: when A, Δx, and Δt are given, the P, Q, R, S, and T are each just a fixed number; The finite difference scheme is actually not as complicated as it appears to be.)
4 points



Useful formula (extracted from Table 6-1 in G&S textbook)

First Derivative		
Method	**Formula**	**Truncation Error**
Two-point forward difference	$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$	$O(h)$
Two-point central difference	$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$	$O(h^2)$

Second Derivative		
Method	**Formula**	**Truncation Error**
Three-point central difference	$f''(x_i) = \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})}{h^2}$	$O(h^2)$

Third Derivative		
Method	**Formula**	**Truncation Error**
Four-point central difference	$f'''(x_i) = \frac{-f(x_{i-2}) + 2f(x_{i-1}) - 2f(x_{i+1}) + f(x_{i+2})}{2h^3}$	$O(h^2)$