MAE 384 HW3

Prob 1 and 2, Sample solutions (Thanks to Brandon Woodward)

1) Solve the eq. using LU decomposition method

$$\begin{bmatrix}3 & 8\\ 6 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}9\\ 4\end{bmatrix} \quad \text{using Crouts} \quad \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = \begin{bmatrix}L_{11} & 0\\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix}1 & U_{12}\\ 0 & 1\end{bmatrix}$$

$i = 1,2 \quad L_{i1} = a_{i1} \Rightarrow L_{11} = a_{11} = 3 \qquad L_{21} = a_{21} = 6$

$$U_{1j} = \frac{a_{1j}}{L_{11}} \Rightarrow U_{12} = \frac{a_{12}}{L_{11}} = \frac{8}{3}$$

$U_{ii} = 1 \Rightarrow U_{11} = U_{22} = 1$

$L_{22} = a_{22} - L_{21}U_{12} \Rightarrow 2 - 6(8/3) = -14$

so, $[a] = [L][U] \Rightarrow \begin{bmatrix}3 & 8\\ 6 & 2\end{bmatrix} = \begin{bmatrix}3 & 0\\ 6 & -14\end{bmatrix}\begin{bmatrix}1 & 8/3\\ 0 & 1\end{bmatrix}$, so

$[L][y] = [b] \Rightarrow \begin{bmatrix}3 & 0\\ 6 & -14\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}9\\ 4\end{bmatrix} \qquad y_1 = \frac{9}{3} = 3$

$y_2 = \frac{4 - 6y_1}{-14} = \frac{4-18}{-14} = 1 \qquad y = \begin{bmatrix}3\\ 1\end{bmatrix}$

And finally, $[u][x] = [y]$

$\Rightarrow \begin{bmatrix}1 & 8/3\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}3\\ 1\end{bmatrix} \qquad x_2 = 1$

$x_1 = 3 - 8/3(\overset{1}{x_2}) \Rightarrow 1/3$

$$\boxed{x = \begin{bmatrix}1/3\\ 1\end{bmatrix}}$$

2) Solve using Gauss elimination method

$$\begin{bmatrix}0 & 1 & 3\\ 2 & 3 & -1\\ -1 & -1 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}9\\ 11\\ 5\end{bmatrix} \Rightarrow \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ 0 & a'_{22} & a'_{23}\\ 0 & 0 & a''_{33}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}b_1\\ b'_2\\ b'_3\end{bmatrix}$$

Swap rows 1 + 3 $\Rightarrow \left[\begin{array}{ccc|c}-1 & -1 & 5 & 5\\ 2 & 3 & -1 & 11\\ 0 & 1 & 3 & 9\end{array}\right]$ $\quad m_{21} = \frac{a_{21}}{a_{11}} = -2$

$a'_{22} = a_{22} - a_{12}(m_{21}) = 3 - (+2) = 1$

$a'_{23} = a_{23} - a_{13}(m_{21}) = -1 - (-10) = 9$

$b'_2 = b_2 - b_1(m_{21}) = 11 - (-10) = 21$

$m_{32} = \frac{a_{32}}{a_{22}'} = \frac{1}{1} = 1$

$a'_{33} = a_{33} - m_{32}(a_{23}') = 3 - 1(9) = -6$

$b'_3 = b_3 - b'_2(m_{32}) = 9 - 1(21) = -12$

$\Rightarrow \begin{bmatrix}-1 & -1 & 5\\ 0 & 1 & 9\\ 0 & 0 & -6\end{bmatrix}\begin{bmatrix}x_3\\ x_2\\ x_1\end{bmatrix} = \begin{bmatrix}5\\ 21\\ -12\end{bmatrix}$

$-6x_1 = -12 \qquad x_1 = 2$

$x_2 + 9x_1 = 21 \qquad x_2 = 3$

$-x_3 - 3 + 10 = 5 \qquad x_3 = 2$

$$\boxed{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}2\\ 3\\ 2\end{bmatrix}}$$

Prob 3 and 4, Sample solutions (Thanks to Jaya Singh)

Prob 3

3. Find C -- the Condition Number of the matrix: (using the Euclidean norm)

$$A = \begin{pmatrix} 6 & 1 \\ 2 & 7 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 7/40 & -1/40 \\ -1/20 & 3/20 \end{pmatrix}$$

Euclidean norm:

$$\|A\|_2 = \sqrt{(6)^2 + (1)^2 + (2)^2 + (7)^2}$$

$$= \sqrt{36 + 1 + 4 + 49}$$

$$= \sqrt{90} = 9.48683$$

$$\|A^{-1}\|_2 = \sqrt{\left(7/40\right)^2 + \left(-1/40\right)^2 + \left(-1/20\right)^2 + \left(3/20\right)^2}$$

$$= 0.23717$$

Condition Number: $C = \|A\|_2 \cdot \|A^{-1}\|_2$

$$= \boxed{2.25}$$

(b) Given that the value of the Condition Number, 'C,' is relatively small $\Rightarrow$ the problem: $\underline{\underline{A}}\,\underline{x} + \underline{b}$ is NOT ill-conditioned.

(Prob 4 next page)

Prob 4

4. Solve the system of linear equations using the Gauss-Siedel Iteration Method;

$$6x_1 + x_2 + x_3 = 4$$
$$x_1 + 6x_2 + x_3 = 4$$
$$x_1 + x_2 + 6x_3 = 4$$

Initial Guess: $\begin{pmatrix} x_1^{I} \\ x_2^{I} \\ x_3^{I} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow$ Given

$$\begin{pmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \xrightarrow{\text{divide by diagonal}} \begin{aligned} x_1 &= 4/6 - 1/6\, x_2 - 1/6\, x_3 \\ x_2 &= 4/6 - 1/6\, x_1 - 1/6\, x_3 \\ x_3 &= 4/6 - 1/6\, x_1 - 1/6\, x_2 \end{aligned}$$

$$\hookrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} - \begin{pmatrix} 0 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Gauss-Siedel Method:

$$x_1^{II} = 2/3 - 1/6\, x_2^{I} - 1/6\, x_3^{I} = 2/3 - 1/6\,(0) - 1/6\,(0) = 2/3$$

$$x_2^{II} = 2/3 - 1/6\, x_1^{II} - 1/6\, x_3^{I} = 2/3 - 1/6\,(2/3) - 1/6\,(0) = 5/9$$

$$x_3^{II} = 2/3 - 1/6\, x_1^{II} - 1/6\, x_2^{II} = 2/3 - 1/6\,(2/3) - 1/6\,(5/9) = 25/54$$

$$x_1^{III} = 2/3 - 1/6\, x_2^{II} - 1/6\, x_3^{II} = 2/3 - 1/6\,(5/9) - 1/6\,(25/54) = 161/324$$

$$x_2^{III} = 2/3 - 1/6\, x_1^{III} - 1/6\, x_3^{II} = 2/3 - 1/6\,(161/324) - 1/6\,(25/54) = 985/1944$$

$$x_3^{III} = 2/3 - 1/6\, x_1^{III} - 1/6\, x_2^{III} = 2/3 - 1/6\,(161/324) - 1/6\,(985/1944) = 0.4994$$

$$x_1^{IV} = 2/3 - 1/6\, x_2^{III} - 1/6\, x_3^{III} = 2/3 - 1/6\,(985/1944) - 1/6\,(0.4994) = 0.49898$$

$$x_2^{IV} = 2/3 - 1/6\, x_1^{IV} - 1/6\, x_3^{III} = 2/3 - 1/6\,(0.49898) - 1/6\,(0.4994) = 0.50027$$

$$x_3^{IV} = 2/3 - 1/6\, x_1^{IV} - 1/6\, x_2^{IV} = 2/3 - 1/6\,(0.49898) - 1/6\,(0.50027) = 0.50012$$

Iteration I: $\begin{pmatrix} x_1^{II} \\ x_2^{II} \\ x_3^{II} \end{pmatrix} = \begin{pmatrix} 2/3 \\ 5/9 \\ 25/54 \end{pmatrix}$

Iteration II: $\begin{pmatrix} x_1^{III} \\ x_2^{III} \\ x_3^{III} \end{pmatrix} = \begin{pmatrix} 161/324 \\ 985/1944 \\ 0.4994 \end{pmatrix}$

Iteration III: $\begin{pmatrix} x_1^{IV} \\ x_2^{IV} \\ x_3^{IV} \end{pmatrix} = \begin{pmatrix} 0.49898 \\ 0.50027 \\ 0.50012 \end{pmatrix}$