MAE384 Homework 4

Discussion

Prob 1

Note that the Lagrange function, $L_i(x)$, should satisfy

$$L_i(x) = 0, \text{ at } x = x_j , j \neq i , \text{ and}$$
$$L_i(x) = 1, \text{ at } x = x_i$$

For example, L_1, L_3, and L_4 should all be zero at $x = x_2 = 1$, while $L_2 = 1$ at $x = x_2$. See sample solution. Some have chosen to plot $y_iL_i(x)$ instead of $L_i(x)$. In this case, each of the $y_iL_i(x)$ curve should pass one of the data points. See additional figure in sample solution.

Prob 2

Two sample solutions are attached for this problem.

Note that the slope of the spline should be continuous at the knots. If the solution is incorrect it would not satisfy this requirement. One can immediately check this property by inspecting the plot of the splines.

Further remark:

While most people have solved Prob 2 by solving the 9x9 (or 8x8 if one eliminates the coefficient a_1 from the outset) system of linear equations using Matlab, two of you have actually solved it by hand. It turned out to be not so complicated. See the nice demonstration in the alternative sample solution. The linear system is relatively easy to solve because it is governed by a *sparse matrix* - matrix with a lot of zero elements. Another important type of sparse matrix is *tridiagonal matrix* (see Section 4.9 in textbook) which we will soon encounter in the numerical solutions for boundary value problems in Ch. 9.

Sample solution, Prob 1 (Thanks to Jarad Hawk)

1

X	0	1	2	3
Y	1	6	25	76

$$y = ax^3 + bx^2 + cx + d$$

$$L_1 = \begin{cases} 1 & x = x_1 \\ 0 & x = x_2 \\ 0 & x = x_3 \\ 0 & x = x_4 \end{cases} \quad L_2 = \begin{cases} 0 & x = x_1 \\ 1 & x = x_2 \\ 0 & x = x_3 \\ 0 & x = x_4 \end{cases} \quad L_3 = \begin{cases} 0 & x = x_1 \\ 0 & x = x_2 \\ 1 & x = x_3 \\ 0 & x = x_4 \end{cases} \quad L_4 = \begin{cases} 0 & x = x_1 \\ 0 & x = x_2 \\ 0 & x = x_3 \\ 1 & x = x_4 \end{cases}$$

$$L_1(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = \frac{(x^2-3x+2)(x-3)}{-6}$$

$$\boxed{L_1(x) = \frac{x^3-6x^2+11x-6}{-6}}$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{(x^2-2x)(x-3)}{2}$$

$$\boxed{L_2(x) = \frac{x^3-5x^2+6x}{2}}$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} = \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = \frac{(x^2-x)(x-3)}{-2}$$

$$\boxed{L_3(x) = \frac{x^3-4x^2+3x}{-2}}$$

$$L_4(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{(x^2-x)(x-2)}{6}$$

$$\boxed{L_4(x) = \frac{x^3-3x^2+2x}{6}}$$

$$y = y_1L_1(x) + y_2L_2(x) + y_3L_3(x) + y_4L_4(x)$$

$$y = 1\left(\frac{x^3-6x^2+11x-6}{-6}\right) + 6\left(\frac{x^3-5x^2+6x}{2}\right) + 25\left(\frac{x^3-4x^2+3x}{-2}\right) + 76\left(\frac{x^3-3x^2+2x}{6}\right)$$

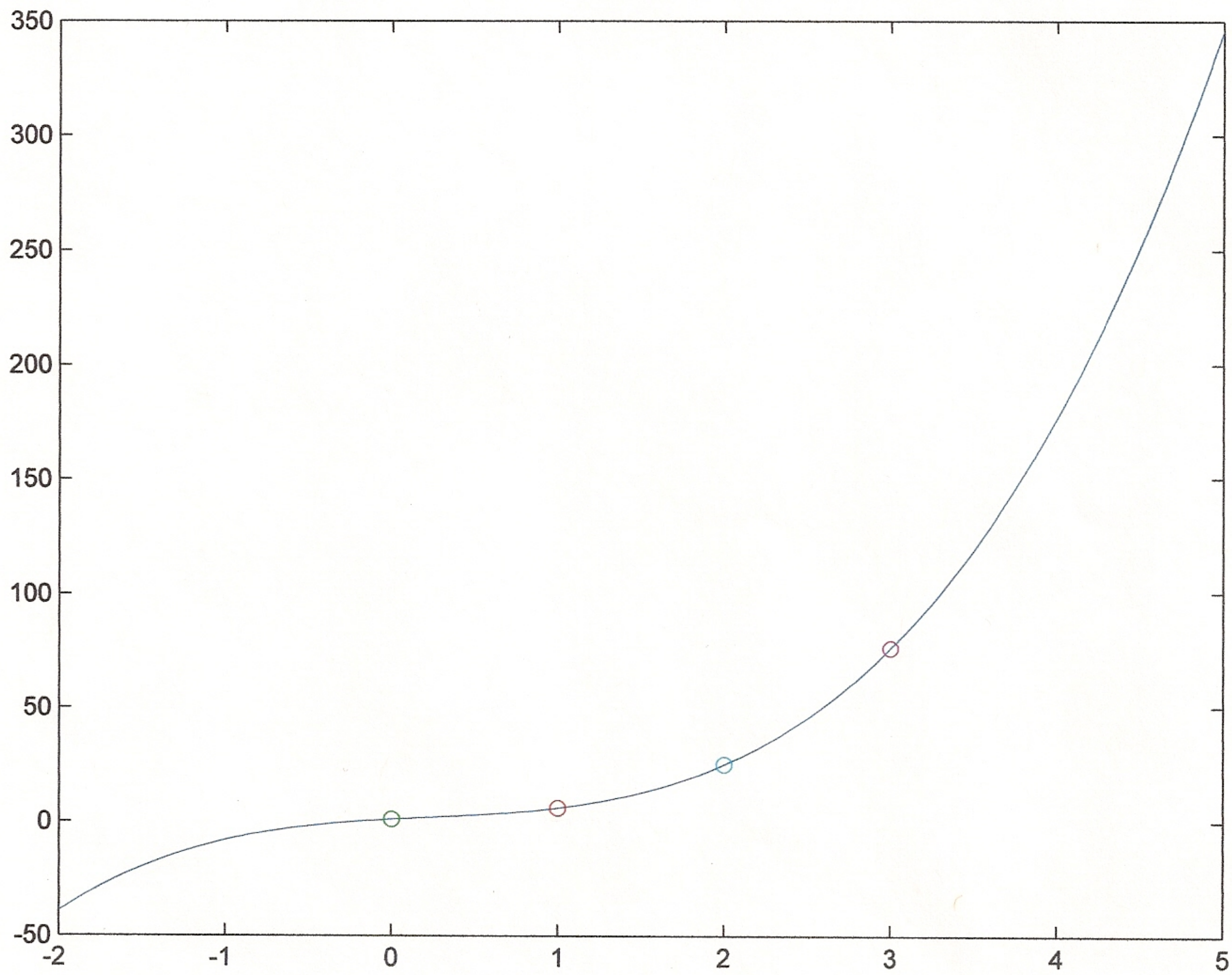
$$\boxed{y = 3x^3 - 2x^2 + 4x + 1}$$

$$f(1.5) = 3(1.5)^3 - 2(1.5)^2 + 4(1.5) + 1$$

$$\boxed{f(1.5) = 12.625}$$

(Continued)

(Prob 1 continued)



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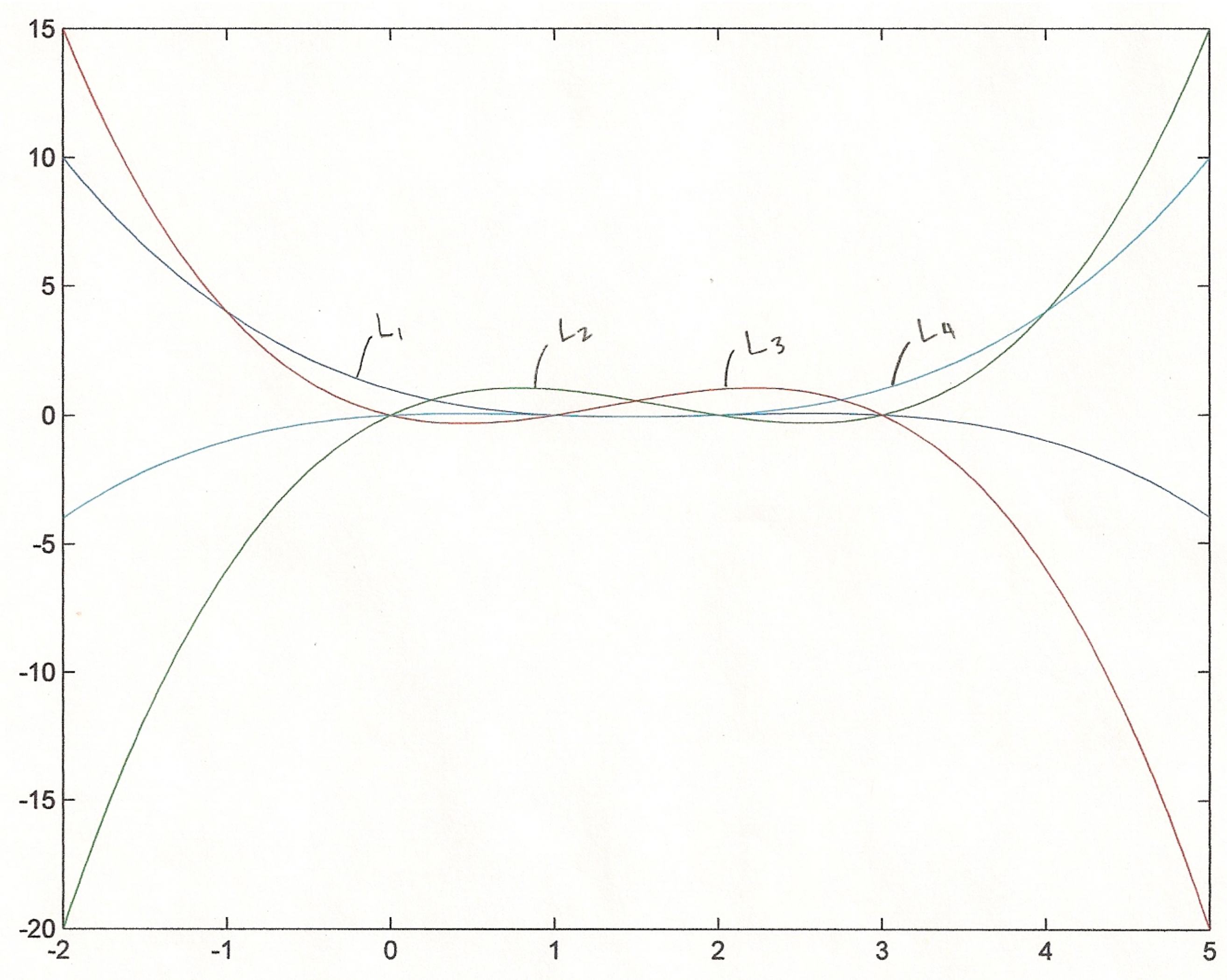
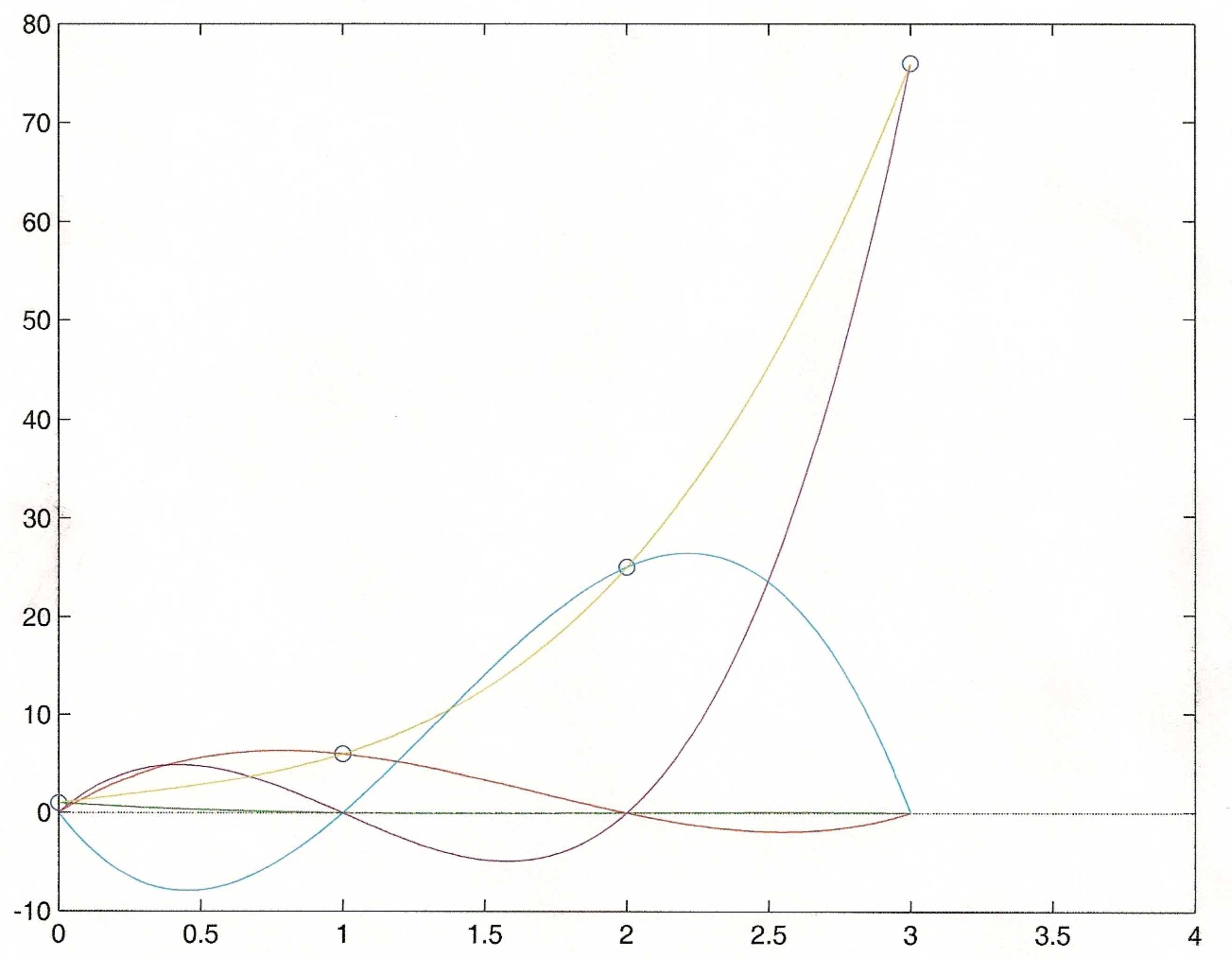
(Prob 1 continued; This is the plot for Lagrange function $L_i(x)$)

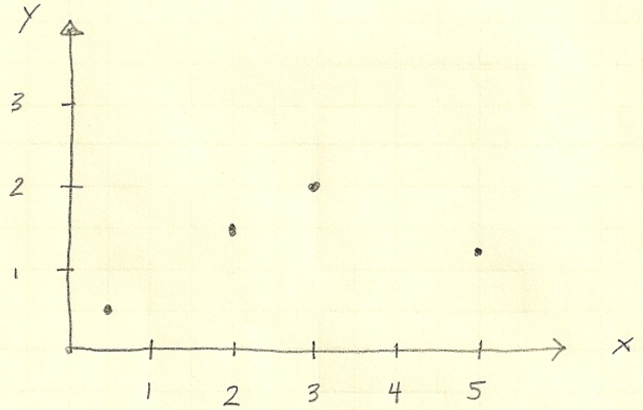


Alternative plot for Prob 1; This is the plot for $y_i L_i(x)$ along with the original data points and the 3rd order polynomial (Thanks to Mike MacDonald)



Sample solution (version 1), Prob 2 (Thanks to Mike MacDonald)

a. Determine the quadratic splines that fit the data.



Interval I : $Y = a_1 x^2 + b_1 x + c_1$

at $\begin{pmatrix} X = x_1 \\ Y = Y_1 \end{pmatrix}$: $Y_1 = a_1 x_1^2 + b_1 x_1 + c_1 \rightarrow (0.5)^2 a_1 + 0.5 b_1 + c_1 = 0.5$

$0.25 a_1 + 0.5 b_1 + c_1 = 0.5$ (i)

at $\begin{pmatrix} X = x_2 \\ Y = Y_2 \end{pmatrix}$: $Y_2 = a_1 x_2^2 + b_1 x_2 + c_1 \rightarrow (2)^2 a_1 + 2b_1 + c_1 = 1.5$

$4a_1 + 2b_1 + c_1 = 1.5$ (ii)

Interval II : $Y = a_2 x^2 + b_2 x + c_2$

at $\begin{pmatrix} x = x_2 \\ Y = Y_2 \end{pmatrix}$: $Y_2 = a_2 x_2^2 + b_2 x_2 + c_2 \rightarrow (2)^2 a_2 + 2b_2 + c_2 = 1.5$

$4a_2 + 2b_2 + c_2 = 1.5$ (iii)

at $\begin{pmatrix} X = x_3 \\ Y = Y_3 \end{pmatrix}$: $Y_3 = a_2 x_3^2 + b_2 x_3 + c_2 \rightarrow (3)^2 a_2 + 3b_2 + c_2 = 2$

$9a_2 + 3b_2 + c_2 = 2$ (iv)

(Continued)

(Prob 2 continued)

2. Interval III : $y = a_3 x^2 + b_3 x + c_3$

at $\begin{pmatrix} x = x_3 \\ y = y_3 \end{pmatrix}$: $y_3 = a_3 x_3^2 + b_3 x_3 + c_3 \rightarrow (3)^2 a_3 + 3b_3 + c_3 = 2$

$$9a_3 + 3b_3 + c_3 = 2 \quad (v)$$

at $\begin{pmatrix} x = x_4 \\ y = y_4 \end{pmatrix}$ $y_4 = a_3 x_4^2 + b_3 x_4 + c_3 \rightarrow (5)^2 a_3 + 5b_3 + c_3 = 1.2$

$$25a_3 + 5b_3 + c_3 = 1.2 \quad (vi)$$

At 1st knot :

$y'_{I} = 2a_1 x_2 + b_1 \qquad y'_{II} = 2a_2 x_2 + b_2$

$$2a_1 x_2 + b_1 = 2a_2 x_2 + b_2 \rightarrow 4a_1 + b_1 = 4a_2 + b_2$$

$$4a_1 + b_1 - 4a_2 - b_2 = 0 \quad (vii)$$

Final Condition: At 1st Point

$y''_{I} = 0 \rightarrow 2a_1 = 0 \quad (ix)$

At 2nd knot :

$y'_{II} = 2a_2 x_3 + b_2 \qquad y'_{III} = 2a_3 x_3 + b_3$

$$2a_2 x_3 + b_2 = 2a_3 x_3 + b_3 \rightarrow 6a_2 + b_2 = 6a_3 + b_3$$

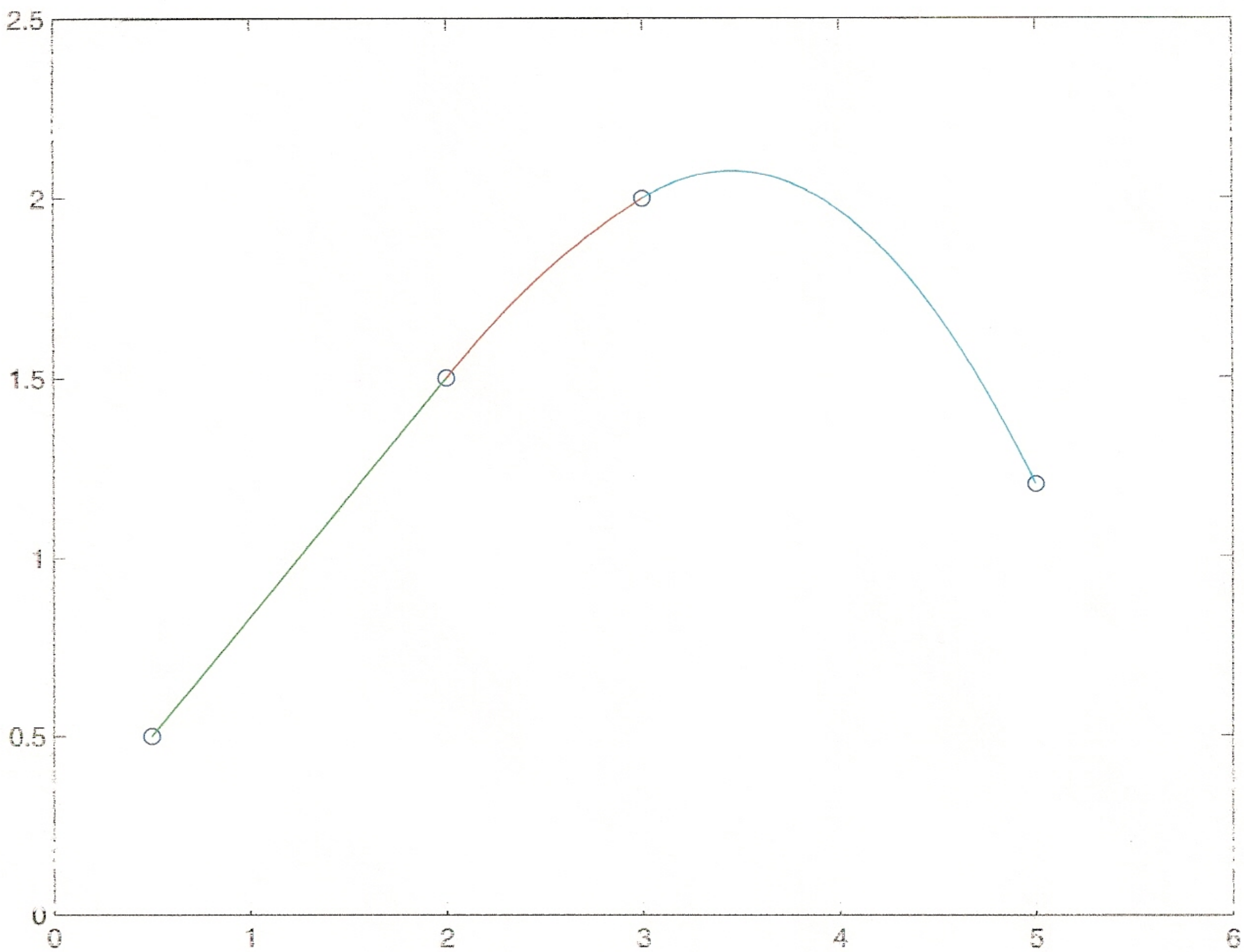
$$0a_1 + 0b_1 + 6a_2 + 1b_2 - 6a_3 - b_3 = 0 \quad (viii)$$

$$\underset{d}{\begin{bmatrix} 0.25 & 0.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 25 & 5 & 1 \\ 4 & 1 & 0 & -4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 1 & 0 & -6 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \underset{e}{\begin{bmatrix} 0.5 \\ 1.5 \\ 1.5 \\ 2 \\ 2 \\ 1.2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

(Continued)

(Prob 2 continued)

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \\ 1/6 \\ -1/6 \\ 4/3 \\ -1/2 \\ -11/30 \\ 38/15 \\ -23/10 \end{bmatrix}$$



Alternative sample solution, Prob 2 (Thanks to Micah Ellis)

a.

x	0.5	2	3	5
y	0.5	1.5	2	1.2

$f_i(x) = a_i x^2 + b_i x + c_i$

$f_1''(x) = 0 = 2a_1 \Rightarrow a_1 = 0$ --- (1)

$\underline{i=1}$: $a_1(.5)^2 + b_1(.5) + c_1 = 0.5 \Rightarrow .5b_1 + c_1 = 0.5$ --- (2)

$a_1(2)^2 + b_1(2) + c_1 = 1.5 \Rightarrow 2b_1 + c_1 = 1.5$ --- (3)

(2), (3) $\Rightarrow b_1 = 2/3,\ c_1 = 1/6$

$\underline{i=2}$: $a_2(2)^2 + b_2(2) + c_2 = 1.5$ --- (4)

$a_2(3)^2 + b_2(3) + c_2 = 2$ --- (5)

$\frac{df_1(x_2)}{dx_2} = \frac{df_2(x_2)}{dx_2} \Rightarrow 2a_1(2) + b_1 = 2a_2(2) + b_2$

$\Rightarrow 2/3 = 4a_2 + b_2$ --- (6)

(4), (5), (6) $\Rightarrow a_2 = -1/6,\ b_2 = 4/3,\ c_2 = -1/2$

$\underline{i=3}$: $a_3(3)^2 + b_3(3) + c_3 = 2$ --- (7)

$a_3(5)^2 + b_3(5) + c_3 = 1.2$ --- (8)

$\frac{df_2(x_3)}{dx_3} = \frac{df_3(x_3)}{dx_3} \Rightarrow 2a_2(3) + b_2 = 2a_3(3) + b_3$

$\Rightarrow 1/3 = 6a_3 + b_3$ --- (9)

(7)(8)(9) $\Rightarrow a_3 = -11/30,\ b_3 = 38/15,\ c_3 = -23/10$

$\therefore$ 3 Quadradic Splines:

$0.5 < x < 2: \quad f_1(x) = \frac{2}{3}x + \frac{1}{6}$

$2 < x < 3: \quad f_2(x) = -1/6\,x^2 + 4/3\,x - 1/2$

$3 < x < 5: \quad f_3(x) = -\frac{11}{30}x^2 + \frac{38}{15}x - \frac{23}{10}$