Sample solution, Prob 1(a)

Prob 1 (a)

$$\frac{du}{dx} = f(x,u), \quad f(x,u) \equiv -u^2 + \sin x, \quad u(0) = 0 \Rightarrow u_0 = 0$$

Euler's explicit method, $\Delta x = 0.1$:

$x=0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad \underline{\underline{x_4}}$

$\Delta x = 0.1$

$$u_1 = \cancelto{0}{u_0} + f(x_0, u_0)\Delta x$$
$$= \left(-\cancelto{0}{u^2(0)} + \cancelto{0}{\sin(0)}\right)\cdot(0.1) = 0$$

$$u_2 = u_1 + f(x_1, u_1)\cdot\Delta x$$
$$= 0 + \left(-u_1^2 + \sin(x_1)\right)\cdot(0.1)$$
$$= 0 + (0 + \sin(0.1))\cdot(0.1) = 0.00998334$$

$$u_3 = u_2 + f(x_2, u_2)\cdot\Delta x$$
$$= 0.00998334 + \left(-(0.00998334)^2 + \sin(0.2)\right)\cdot(0.1)$$
$$= 0.0298403$$

$$u_4 = u_3 + f(x_3, u_3)\cdot\Delta x$$
$$= 0.0298403 + \left(-(0.0298403)^2 + \sin(0.3)\right)\cdot(0.1)$$
$$= \boxed{0.0593032}$$

#

Sample solution, Prob 1(b)

$$\frac{du}{dx} = f(x,u), \quad f(x,u) \equiv -u^2 + \sin x, \quad u(0) = 0$$

4th order R-K method; $\Delta x = 0.2$:

$x = 0$ 0.2 0.4

x_0 x_1 x_2

$\Delta x = 0.2$

$$u_1 = u_0 + \frac{\Delta x}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

where $K_1 = f(x_0, u_0)$

$$= -u_0^2 + \sin x_0 = -u^2(0) + \sin(0) = 0$$

$$K_2 = f\left(x_0 + \tfrac{1}{2}\Delta x,\ u_0 + \tfrac{1}{2}K_1 \cdot \Delta x\right)$$

$$= -\left(u_0 + \tfrac{1}{2}K_1\Delta x\right)^2 + \sin\left(x_0 + \tfrac{1}{2}\Delta x\right)$$

$$= -\left(0 + \cancel{\tfrac{1}{2}\,0\cdot(0.2)}^{\,0}\right)^2 + \sin\left(0 + \tfrac{1}{2}\cdot 0.2\right)$$

$$= \sin(0.1) = 0.0998334$$

$$K_3 = f\left(x_0 + \tfrac{1}{2}\Delta x,\ u_0 + \tfrac{1}{2}K_2 \cdot \Delta x\right)$$

$$= -\left(u_0 + \tfrac{1}{2}K_2\Delta x\right)^2 + \sin\left(x_0 + \tfrac{1}{2}\Delta x\right)$$

$$= -\left[\tfrac{1}{2}\cdot(0.09983)\cdot(0.2)\right]^2 + \sin(0.1)$$

$$= -0.00009966 + 0.0998334$$

$$= 0.0997337$$

$$K_4 = f\left(x_0 + \Delta x,\ u_0 + K_3\Delta x\right)$$

$$= -\left(u_0 + K_3\Delta x\right)^2 + \sin\left(x_0 + \Delta x\right)$$

$$= -\left(0.0997337 \times 0.2\right)^2 + \sin(0.2)$$

$$= -0.00039787 + 0.1986693$$

$$= 0.1982714$$

$$u_1 = \cancel{u_0}^{\,0} + \frac{0.2}{6}\left[0 + 2\cdot 0.0998334 + 2\cdot 0.0997337 + 0.1982714\right]$$

$$= 0.0199135$$

(Prob 1(b) continued to next page)

(Prob 1(b) continued)

$$u_2 = u_1 + \frac{\Delta x}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

where

$$K_1 = f(x_1, u_1) = -u_1^2 + \sin(x_1)$$
$$= \underbrace{-(0.0199135)^2}_{-0.0003965474} + \underbrace{\sin(0.2)}_{0.19866933} = 0.19827278$$

$$K_2 = f(x_1 + \tfrac{1}{2}\Delta x,\ u_1 + \tfrac{1}{2}K_1\Delta x)$$
$$= -(u_1 + \tfrac{1}{2}K_1\Delta x)^2 + \sin(\overbrace{x_1 + \tfrac{1}{2}\Delta x}^{0.3})$$
$$= -(0.0199135 + \tfrac{1}{2}\cdot 0.19827278 \cdot 0.2)^2 + \sin(0.2 + \tfrac{1}{2}\cdot 0.2)$$
$$= -0.0015793294 + 0.295520206$$
$$= 0.29394087$$

$$K_3 = f(x_1 + \tfrac{1}{2}\Delta x,\ u_1 + \tfrac{1}{2}K_2\Delta x)$$
$$= -(u_1 + \tfrac{1}{2}K_2\Delta x)^2 + \sin(x_1 + \tfrac{1}{2}\Delta x)$$
$$= -(0.0199135 + \tfrac{1}{2}\cdot(0.2939408)\cdot(0.2))^2 + \sin(0.3)$$
$$= -0.0024312374 + 0.295520206$$
$$= 0.293088969$$

$$K_4 = f(x_1 + \Delta x,\ u_1 + K_3\Delta x)$$
$$= -(u_1 + K_3\Delta x)^2 + \sin(x_1 + \Delta x)$$
$$= -(0.0199135 + (0.293088969)\cdot(0.2))^2 + \overbrace{\sin(0.4)}^{0.389418342}$$
$$= -0.0061671641 + 0.389418342$$
$$= 0.38325117$$

$$u_2 = u_1 + \frac{0.2}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$
$$= 0.0199135 + \frac{0.2}{6}[0.19827278 + 2\times 0.29394087 + 2\times 0.293088969 + 0.38325117]$$

$$= \boxed{0.07843295}$$

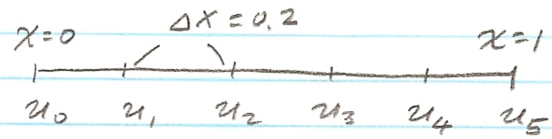
#

Sample solution, Prob 2(a)

Prob 2 (a) Numerical solution

$$\frac{d^2u}{dx^2} - 3\frac{du}{dx} + 2u = 0 \qquad u(0) = 0 \quad u(1) = 1 - e$$

$$\frac{d^2u}{dx^2} \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2}$$



$$\frac{du}{dx} \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

$$\Rightarrow \quad \frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2} - 3\,\frac{u_{i+1} - u_{i-1}}{2\Delta x} + 2u_i = 0$$

$$\Rightarrow \quad \left(\frac{1}{(\Delta x)^2} + \frac{3}{2\Delta x}\right) u_{i-1} + \left(2 - \frac{2}{(\Delta x)^2}\right) u_i + \left(\frac{1}{(\Delta x)^2} - \frac{3}{2\Delta x}\right) u_{i+1} = 0$$

since $\Delta x = 0.2$, we have

$$32.5\, u_{i-1} - 48\, u_i + 17.5\, u_{i+1} = 0$$

(Or, you can multiply a constant to the above equation if you wish)

$$\Rightarrow \quad \begin{aligned}
32.5\,\underbrace{\boxed{u_0}}_{0} - 48\,u_1 + 17.5\,u_2 &= 0 \\
32.5\,u_1 - 48\,u_2 + 17.5\,u_3 &= 0 \\
32.5\,u_2 - 48\,u_3 + 17.5\,u_4 &= 0 \\
32.5\,u_3 - 48\,u_4 + 17.5\,\underbrace{\boxed{u_5}}_{1-e} &= 0
\end{aligned}$$

$$\Rightarrow \quad \begin{pmatrix} -48 & 17.5 & 0 & 0 \\ 32.5 & -48 & 17.5 & 0 \\ 0 & 32.5 & -48 & 17.5 \\ 0 & 0 & 32.5 & -48 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -17.5(1-e) \end{pmatrix}$$

Solving it, we obtain numerical solution;

$$\boxed{\begin{gathered}(u_1, u_2, u_3, u_4) = (-0.094759, -0.259911, -0.536916, -0.989994) \\ \text{along with } u_0 = 0 \text{ and } u_5 = 1 - e = -1.71828\end{gathered}}$$

#

Sample solution, Prob 2(b), analytic solution + remarks

$$\frac{d^2u}{dx^2} - 3\frac{du}{dx} + 2u = 0 \qquad u(0) = 0 \qquad u(1) = 1 - e$$

assume $u \propto e^{\alpha x}$

$\Rightarrow \quad \alpha^2 - 3\alpha + 2 = 0 \quad \Rightarrow \quad \alpha = 1, 2$

$\Rightarrow \quad u(x) = A e^{x} + B e^{2x}$

using boundary condition $u(0) = 0 \Rightarrow A + B = 0 \quad \underline{\underline{B = -A}}$

using boundary condition $u(1) = 1 - e$

$\Rightarrow \quad A e - A e^2 = 1 - e \quad \Rightarrow \quad \underline{A = e^{-1}} \quad \Rightarrow \underline{B = -e^{-1}}$

$\Rightarrow \quad u(x) = e^{-1} e^{x} - e^{-1} e^{2x},$

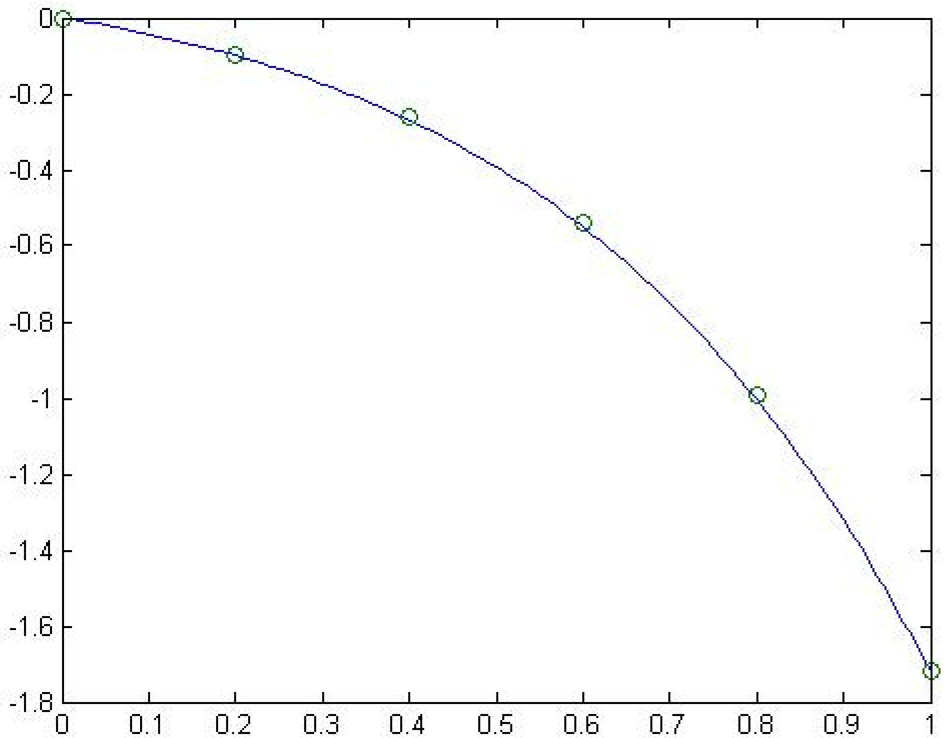
or $\boxed{u(x) = e^{x-1} - e^{2x-1}}$ #

In the attached plot, analytic solution is the solid curve. Numerical solution is in circle.

#

* Note: The matrix for the numerical solution in Part (a) is not unique. If you wish, you can multiply the matrix [A], and the vector in the right hand side [b], by a constant and the solution is still the same.

Problem 2(b) plot. The abscissa is x. Solid curve = analytic colution; Circles = numerical solution



Sample solution, Prob 3

Solve

$$y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = 0$$

let $u(x,y) = G(x)\,H(y)$

$$\Rightarrow \quad yH \frac{dG}{dx} + xG \frac{dH}{dy} = 0$$

$$\Rightarrow \quad \frac{1}{xG}\frac{dG}{dx} = -\frac{1}{yH}\frac{dH}{dy} = c$$

$$\begin{cases} \dfrac{1}{xG}\dfrac{dG}{dx} = c & \text{--- (1)} \\[2ex] \dfrac{1}{yH}\dfrac{dH}{dy} = -c & \text{--- (2)} \end{cases}$$

This is (unimportant) $k_1 = e^{k_1^*}$ ~~const~~

From (1): $\frac{1}{G}\frac{dG}{dx} = cx \qquad \frac{d\ln G}{dx} = cx$

$$\Rightarrow \ln G = \frac{c}{2}x^2 + k_1^* \Rightarrow G(x) = k_1 e^{\frac{c}{2}x^2}$$

Similarly, from (2): $H(y) = k_2 e^{-\frac{c}{2}y^2}$

$$\Rightarrow \boxed{u(x,y) = G(x)H(y) = k\, e^{\frac{c}{2}(x^2 - y^2)}}$$

#

Sample solution, Prob 4

$$\frac{\partial u}{\partial t} = A\frac{\partial^3 u}{\partial x^3} - 0.1\,u$$

$$\frac{\partial u}{\partial t} \approx \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

$$\frac{\partial^3 u}{\partial x^3} \approx \frac{-u_{i-2,j} + 2\,u_{i-1,j} - 2\,u_{i+1,j} + u_{i+2,j}}{2\,(\Delta x)^3}$$

$\Rightarrow$

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = A\,\frac{-u_{i-2,j} + 2u_{i-1,j} - 2\,u_{i+1,j} + u_{i+2,j}}{2\,(\Delta x)^3} - 0.1\,u_{i,j}$$

$\Rightarrow$

$$u_{i,j+1} = -\left(\frac{A\Delta t}{2(\Delta x)^3}\right)u_{i-2,j} + 2\left(\frac{A\Delta t}{2(\Delta x)^3}\right)u_{i-1,j} + (1-0.1\,\Delta t)\,u_{i,j}$$

$$- 2\left(\frac{A\Delta t}{2(\Delta x)^3}\right)u_{i+1,j} + \left(\frac{A\Delta t}{2\,(\Delta x)^3}\right)u_{i+2,j}\,,$$

written as:

$$u_{i,j+1} = P\,u_{i-2,j} + Q\,u_{i-1,j} + R\,u_{i,j} + S\,u_{i+1,j} + T\,u_{i+2,j}$$

with $A = 0.8 \quad \Delta x = 0.2 \quad \Delta t = 0.1$:

$$P = -\left(\frac{A\Delta t}{2(\Delta x)^3}\right) = -5$$

$$Q = 2\left(\frac{A\Delta t}{2(\Delta x)^3}\right) = 10$$

$$R = (1 - 0.1\,\Delta t) = 0.99$$

$$S = -2\left(\frac{A\Delta t}{2(\Delta x)^3}\right) = -10$$

$$T = \left(\frac{A\Delta t}{2(\Delta x)^3}\right) = 5$$

(Prob 4 continued to next page)

(Prob 4 continued)

The main equation to use is:

$$\boxed{u_{i,j+1} = -5\,u_{i-2,j} + 10\,u_{i-1,j} + 0.99\,u_{i,j} - 10\,u_{i+1,j} + 5\,u_{i+2,j}}$$

boundary conditions $u_{4,0} = 1$ $u_{5,0} = 0.5$

at $j=1$ $(t=0.1)$:

$u_{2,1} = 5\,u_{4,0} = 5$

$u_{3,1} = -10\,u_{4,0} + 5\,u_{5,0} = -7.5$

$u_{4,1} = 0.99\,u_{4,0} - 10\,u_{5,0} = -4.01$

$u_{5,1} = 10\,u_{4,0} + 0.99\,u_{5,0} = 10.495$

$u_{6,1} = -5\,u_{4,0} + 10\,u_{5,0} = 0$

$u_{7,1} = -5\,u_{5,0} = -2.5$

All other $u_{i,j}$ at $j=1$ are zero.

at $j=2$ $(t=0.2)$

$u_{0,2} = 5\,u_{2,1} = 25$

$u_{1,2} = -10\,u_{2,1} + 5\,u_{3,1} = -87.5$

$u_{2,2} = 0.99\,u_{2,1} - 10\,u_{3,1} + 5\,u_{4,1} = 59.9$

$u_{3,2} = 10\,u_{2,1} + 0.99\,u_{3,1} - 10\,u_{4,1} + 5\,u_{5,1} = 135.15$

$u_{4,2} = -5\,u_{2,1} + 10\,u_{3,1} + 0.99\,u_{4,1} - 10\,u_{5,1} + 5\,u_{6,1} = -208.9199$

$u_{5,2} = -5\,u_{3,1} + 10\,u_{4,1} + 0.99\,u_{5,1} - 10\,u_{6,1} + 5\,u_{7,1} = -4.70995$

$u_{6,2} = -5\,u_{4,1} + 10\,u_{5,1} + 0.99\,u_{6,1} - 10\,u_{7,1} = 150$

$u_{7,2} = -5\,u_{5,1} + 10\,u_{6,1} + 0.99\,u_{7,1} = -54.95$

$u_{8,2} = -5\,u_{6,1} + 10\,u_{7,1} = -25$

$u_{9,2} = -5\,u_{7,1} = 12.5$

All other $u_{i,j}$ at $j=2$ are zero. #

(see next page for plot at t = 0.2)

Prob 4, plot. The abscissa is the index "*i*". Shown is $u_{i,2}$ for i = 0, 1, 2, ..., 10.