MAE502 Spring 2010 Homework #1

Problem 1 (5 points) (1 point ≈ 1 % of your total score for the course)

(a) Solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

for $u(x,t)$ defined within the domain of $0 \le x \le 1$ and $t \ge 0$, given the following boundary conditions

(i) $u_x(0, t) = 0$ (Note: $u_x \equiv \partial u/\partial x$)
(ii) $u(1, t) = 0$
(iii) $u(x,0) = F(x)$, where $F(x) = 2 \cos(0.5\, \pi x) + \cos(3.5\, \pi x)$.

Plot the solution as a function of x at $t = 0$ (the "initial state"), 0.004, 0.01, 0.1, and 0.5.

> The above PDE describes the evolution of the temperature, u(x,t), within a thin metal rod of unit length ($0 \le x \le 1$) that is thermally insulated except at its two end points ($x = 0$ and $x = 1$). The 1st b.c. further indicates that the left end of that metal rod is also thermally insulated, allowing no heat flux to go through it. The 2nd b.c. demands that the temperature at the right end point be fixed to 0, which may be achieved by attaching that end of the rod to a thermal bath with u = 0. The third b.c. prescribes the initial distribution of temperature along the metal rod.

(b) What is the equilibrium solution for the system in (a)? Does your solution in (a) approach the equilibrium solution at a large t?

(c) At $x = 1$, temperature is fixed but heat flux is allowed to change. Based on your solution in (a), calculate and plot the heat flux, $\phi \equiv -\,\partial u/\partial x$, at $x = 1$ as a function of t. (Note that this plot would track the rate of external heat gain or loss for the whole metal rod by the heat flow through its right end, as one can readily prove that $dE/dt = -\phi(1, t)$, where E is the integral of u over the metal rod.)

(d) The parameter, $S(t) \equiv \sqrt{\int_0^1 (\partial u/\partial x)^2 dx}$, is a useful measure of the sharpness of temperature contrast within the metal rod; A smoother temperature distribution has a smaller value of $S(t)$. Using your solution for u, evaluate and plot $S(t)$ as a function of t.

(e) Using the results from (a)-(d), interpret the behavior of your solution. (If you are paid by a sponsor to solve this heat transfer problem, what story would you be able to tell based on the analytic and numerical results?)

Problem 2 (2 points)
Consider the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

for $u(x,t)$ defined within the domain of $0 \le x \le 1$ and $t \ge 0$, with the following "periodic" boundary conditions,

(i) $u(0, t) = u(1, t)$
(ii) $u_x(0, t) = u_x(1, t)$ (Note: $u_x \equiv \partial u/\partial x$)
(iii) $u(x,0) = F(x)$.

This corresponds to the situation when the metal rod is bent into a ring with its two end points connected to each other. (The left end point now coincides with the right end point; $x = 0$ and $x = 1$ are physically the same point. Therefore, the temperature, u, and heat flux, $-u_x$, at $x = 0$ must equal their counterparts at $x = 1$.) Going through the regular procedure of separation of variables by assuming $u(x,t) = G(x)H(t)$, what would be the eigenvalues and eigenfunctions for the equation for $G(x)$? (You only need to answer this question; No need to obtain the complete solution for the PDE.)

Problem 3 (2 points)
Find the general solution (disregarding boundary conditions) of each of the following PDEs by the method of separation of variables

(i) $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3xu$$

(ii) $$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + u = 0$$