MAE502 Spring 2010 Homework #2

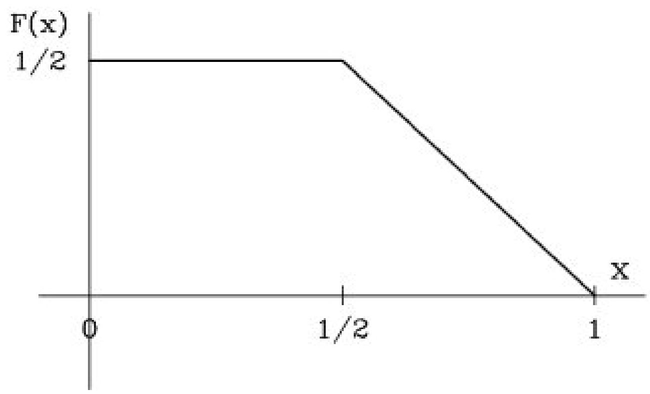
Problem 1 (3 points)
This is a repeat of Prob. 1(a) in Homework #1 but with a more complicated initial distribution of *u*. Solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

for $u(x,t)$ defined within the domain of $0 \le x \le 1$ and $t \ge 0$, given the following boundary conditions

(i) $u_x(0, t) = 0$ (Note: $u_x \equiv \partial u/\partial x$)
(ii) $u(1, t) = 0$
(iii) $u(x,0) = F(x)$, where $F(x) = 1/2$, if $0 \le x \le 1/2$, and $F(x) = (1-x)$, if $1/2 < x \le 1$

See figure below for $F(x)$.



Plot the solution as a function of x, at t = 0.01, 0.1, and 0.5, along with the "initial state", $u(x,0) = F(x)$. It is preferable that all 4 curves are collected in one plot. Discuss the behavior of your solution as $t \to \infty$.

Note: (i) You can recycle the eigenvalues and eigenfunctions obtained from HW1 Prob 1. No need to repeat that part of the process. (ii) The solution of this problem may consist of an infinite series. It is your job to decide where to truncate that series such that the terms you keep are sufficient for constructing an accurate solution. This remark also applies to Problem 2 and to all future homework problems that involve an infinite series.

Problem 2 (5 points)

Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$

for $u(x, y)$ defined within the domain of $0 \le x \le 1$ and $0 \le y \le 1$, given the following boundary conditions,

(i) $u(0, y) = \sin(\pi y) + 0.5 \sin(2\pi y)$
(ii) $u(1, y) = 0$
(iii) $u(x, 0) = 0$
(iv) $u(x,1) = 8\,(x - x^2)$

Make a color/contour plot of your solution in the fashion of Matlab Example #2 or #3 in our course website. What are the values of $u(x, y)$ at (i) $x = 0.5$, $y = 0.5$, and (ii) $x = 0.1$, $y = 0.8$?

• Useful Matlab online documentation: mathworks.com (the maker of Matlab)
→ Products & Services → Product List → MATLAB → Function List
In the "Function List" web page, it is most useful to choose *Alphabetical List* at right.
The Matlab graphics functions that are relevant to this problem are pcolor and contour.

Problem 3 (2 points)
In real world situation, the 1-D Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K \frac{\partial^2 u}{\partial \hat{x}^2} \;,\; 0 \le \hat{x} \le L \quad (L \text{ is the length of the "metal rod", in meters}),$$

where $\hat{t}$ and $\hat{x}$ are time (in seconds) and distance (in meters), K is thermal conductivity in m^2/s, and u is temperature in °C (this last fact is not needed for this problem). In our classroom discussion, we usually consider the non-dimensional form of the equation with u defined on the interval, $0 \le x \le 1$, where x is related to $\hat{x}$ by $\hat{x} = L\,x$. One can also re-scale time by $\hat{t} = T\,t$, where T is a certain dimensional time scale and t is the non-dimensionalized time. In our discussion we also ignored K because (as long as K is constant) the transformation, $x' = x/\sqrt{K}$, or $t' = K\,t$, would eliminate the explicit dependence on K. The combination of these transformations lead to the simpler, nondimensional problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;,\; 0 \le x \le 1 \;.$$

Nevertheless, once the non-dimensional problem is solved, we need to transform the answer back to the real world situation in order for the solution to be useful.

(a) Suppose that the heat equation in our HW1, Prob 1 (in which both x and t are non-dimensional, and $0 \le x \le 1$) describes the real world situation of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem? (We consider $t = 0.01$ because it is about the time when a significant redistribution of temperature begins to take place.)

(b) Same as (a), but suppose that our HW1 Prob 1 describes heat transfer along a wooden stick that is 0.5 meter long and made of pine wood ($K \approx 10^{-7}$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to? Are the time scales you obtained in (a) and (b) consistent with everyday experiences?
(For instance, one can use a long wooden spoon to continuously stir a boiling pot of soup without getting one's hand burned. In contrast, the same practice would make one very uncomfortable if the big spoon is made entirely of copper. Note that the time scale for cooking a pot of soup is about 10 minutes. The length of a big wooden spoon is about a foot, or 0.3 meter.)