Problem 1 (4 points)
This is a repeat of HW2 Prob 2 but we wish to use the numerical (finite difference) method in
Chapter 6 to solve Laplace's equation
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,
\]
for \(u(x, y)\) defined within the domain of \(0 \leq x \leq 1\) and \(0 \leq y \leq 1\), given the following boundary conditions,

(i) \(u(0, y) = \sin(\pi y) + 0.5 \sin(2\pi y)\)
(ii) \(u(1, y) = 0\)
(iii) \(u(x, 0) = 0\)
(iv) \(u(x, 1) = 8(x - x^3)\).

To proceed, discretize the 2nd partial derivatives in both \(x\) and \(y\) by using the 2nd order centered
finite difference scheme (e.g., see Eqs. 6.2.15-6.2.17 in textbook). Then, find the numerical
solutions for \(u(x, y)\) for the two cases with \(\Delta x = 0.1, \Delta y = 0.1\) and \(\Delta x = 0.25, \Delta y = 0.25\). Plot
each of the solutions as a contour and/or color map in the same fashion as you did for the
analytic solution for HW2 Prob 2. For both cases, calculate \(u(x, t)\) at \(x = 0.5, y = 0.5\) and compare
the values with that obtained from the analytic solution in HW2 Prob 2. (Note: Your discretized
equation set will be equivalent to a standard system of linear equations. For a large system,
itative methods such as those in Sec. 6.6 are useful. In our cases, the system is small enough
that a direct solution of the matrix problem using the Matlab operation \(a\backslash b\) will also work.)

Problem 2 (4 points)
(a) Using the method of characteristics, solve the nonlinear (or "quasilinear") PDE,
\[
\frac{\partial u}{\partial t} + 2u \frac{\partial u}{\partial x} = 0,
\]
for \(u(x, t)\) defined on \(-\infty < x < \infty\) and \(0 \leq t < \infty\), given the boundary condition

\[u(x, 0) = P(x),\]
where
\[
P(x) = \begin{cases} 
1 & , x < 0 \\
1 + x^2 & , 0 \leq x \leq 1 \\
2 & , x > 1 
\end{cases}
\]

Using your solution, evaluate \(u(x, t)\) at \((x = 1, t = 0.05)\) and \((x = 0.1, t = 0.1)\).

(b) Plot the solution, \(u(x, t)\), at \(t = 0.1\) and \(t = 0.2\), along with the "initial state", \(u(x, 0) = P(x)\). Briefly discuss the behavior of the solution.