MAE502 Spring 2010 Homework #6

Problem 1 (4 points)
This is a repeat of HW2 Prob 2 but we wish to use the numerical (finite difference) method in Chapter 6 to solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$

for $u(x, y)$ defined within the domain of $0 \le x \le 1$ and $0 \le y \le 1$, given the following boundary conditions,

(i) $u(0, y) = \sin(\pi y) + 0.5 \sin(2\pi y)$
(ii) $u(1, y) = 0$
(iii) $u(x, 0) = 0$
(iv) $u(x,1) = 8\,(x - x^2)$.

To proceed, discretize the 2nd partial derivatives in both x and y by using the 2nd order centered finite difference scheme (e.g., see Eqs. 6.2.15-6.2.17 in textbook). Then, find the numerical solutions for $u(x, y)$ for the two cases with ($\Delta x = 0.1$, $\Delta y = 0.1$) and ($\Delta x = 0.25$, $\Delta y = 0.25$). Plot each of the solutions as a contour and/or color map in the same fashion as you did for the analytic solution for HW2 Prob 2. For both cases, calculate $u(x,t)$ at $x = 0.5$, $y = 0.5$ and compare the values with that obtained from the analytic solution in HW2 Prob 2. (Note: Your discretized equation set will be equivalent to a standard system of linear equations. For a large system, iterative methods such as those in Sec. 6.6 are useful. In our cases, the system is small enough that a direct solution of the matrix problem using the Matlab operation a\b will also work.)

Problem 2 (4 points)
(a) Using the method of characteristics, solve the nonlinear (or "quasilinear") PDE,

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0 \quad ,$$

for $u(x,t)$ defined on $-\infty < x < \infty$ and $0 \le t < \infty$, given the boundary condition

$$u(x,0) = P(x) \; ,$$

where

$$P(x) = \begin{cases} 1 & , \; x < 0 \\ 1 + x^2 & , \; 0 \le x \le 1 \\ 2 & , \; x > 1 \end{cases} \quad .$$

Using your solution, evaluate $u(x,t)$ at ($x = 1$, $t = 0.05$) and ($x = 0.1$, $t = 0.1$).

(b) Plot the solution, $u(x,t)$, at $t = 0.1$ and $t = 0.2$, along with the "initial state", $u(x,0) = P(x)$. Briefly discuss the behavior of the solution.