MAE 502 Partial Differential Equations in Engineering

Spring 2011 Mon/Wed 5:00-6:15 PM

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(Huei rhymes with "way")

Office: **ERC 359**
Office hours: Tuesday 2:00-4:00 PM, or by appointment

Course website

http://www.public.asu.edu/~hhuang38/MAE502.html

- **Updated schedule**
- **Homework assignments/solutions**
- **Supplementary slides**
- **Matlab examples**

Course Outline

(See syllabus)

I. Analytic treatment for linear PDE

1. Overview of PDE
   Commonly encountered PDEs in engineering and science
   Types of PDEs, the physical phenomena they represent, and relevant boundary conditions
2. Method of separation of variables; eigenfunction expansion
3. Fourier Series
   Solution of ODE and PDE by Fourier Series expansion
4. Short review of Sturm-Louville Problem and orthogonal functions;
   Representation using orthogonal basis
5. Fourier transform
   Solution of PDE by Fourier transform; Behavior of solution in spectral space
6. PDE in non-Cartesian geometry
7. Forced problem and brief introduction to Green's function

II. Additional topics

8. Comparison of numerical and analytic methods for Laplace's equation and
   heat equation
9. Very brief introduction to nonlinear PDE
   Examples of nonlinear PDEs for real world phenomena; Behavior of their solutions;
   Conservation laws
10. Method of cheracteristics; Solutions of nonlinear/quasilinear equations.
11. Miscellanies (while time allows)

Textbook:
Applied Partial Differential Equation, by R. Haberman, **Required**

Additional lecture notes/slides will be provided by instructor

Remarks on textbook ...

Grade: Homework 50%
Midterm (one exam) 20%
Final 30%

Matlab (or its equivalent) will be needed for basic computations and plotting of solutions for homework.

Other languages/tools (Fortran, C, C++, Java, etc.) are acceptable, but beware that they may or may not support the graphics functions needed for homework

- ASU students have free access to Matlab through My Apps;
  Choose "2009b" version which is useful enough for our class
  and is relatively glitch-free
  You can access the software using your own laptop, even from home, by logging on to My ASU/My Apps

- Public computers/printers are available through the campus (Noble Library, Goldwater Building, etc.)

- Instructor will provide initial help on Matlab - use office hours

A very short introduction ...

Examples of "classical" linear PDEs

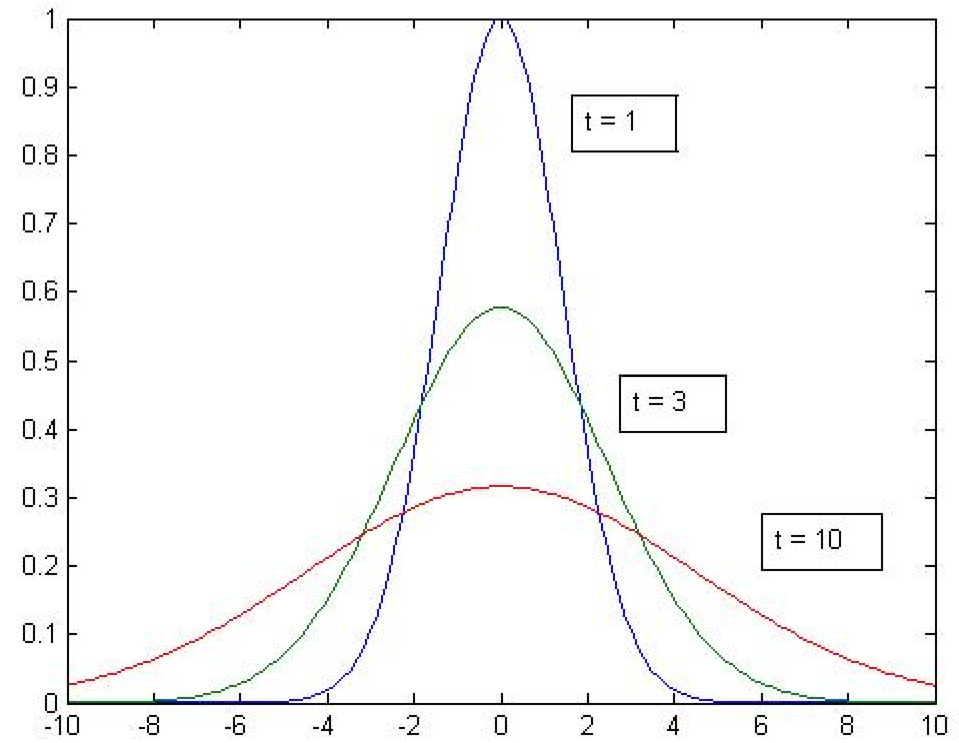
To draw the correspondence between a PDE and a real world phenomenon, we will use t to denote time and (x, y, z) to denote the 3 spatial coordinates

Heat (or diffusion) equation: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *, describes the diffusion of temperature or the density of a chemical constituent from an initially concentrated distribution (e.g., a "hot spot" on a metal rod, or a speck of pollutant in the open air)*

A typical solution (when the initial distribution of u is a "spike"): $u(x,t) = \frac{1}{\sqrt{t}} \exp(-\frac{x^2}{4t})$

(Exercise: Verify that this solution satisfies the PDE)

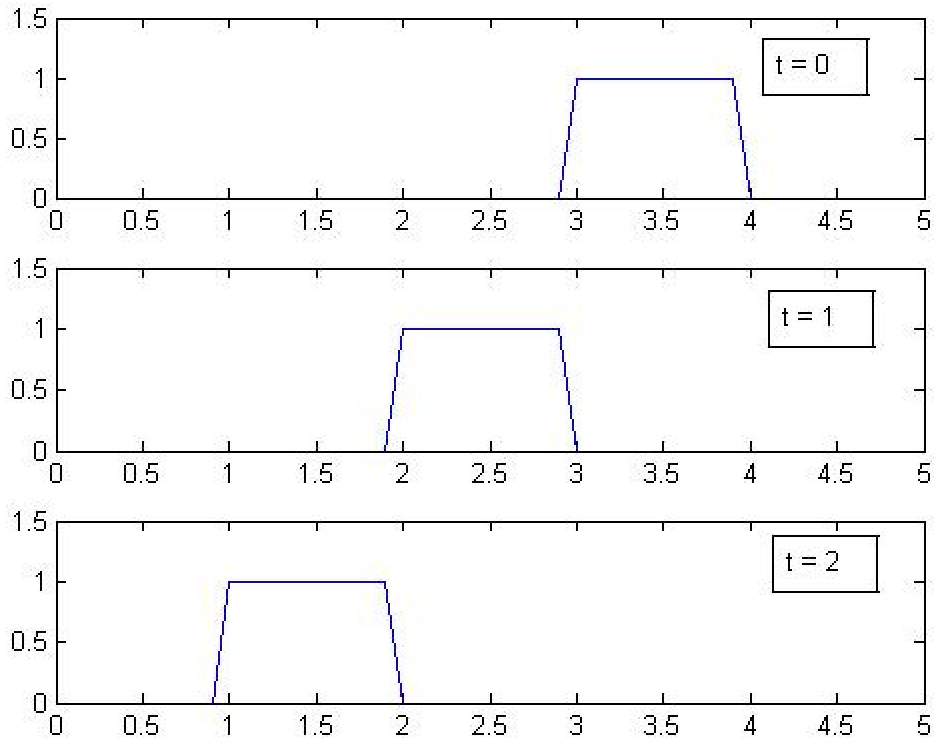
The figure in next page shows this solution at a few different times. As time increases, $u(x)$ becomes broader, its maximum decreases, but its "center of mass" does not move. These features characterize a "diffusion process".



Solution of the heat equation at different times. The three curves are $u(x, 1)$, $u(x, 3)$, and $u(x,10)$

Linear advection equation: $\dfrac{\partial u}{\partial t} = c\,\dfrac{\partial u}{\partial x}$ *, describes the constant movement of an initial distribution of u with a "speed" of – c along the x-axis. The distribution moves while preserving its shape.*

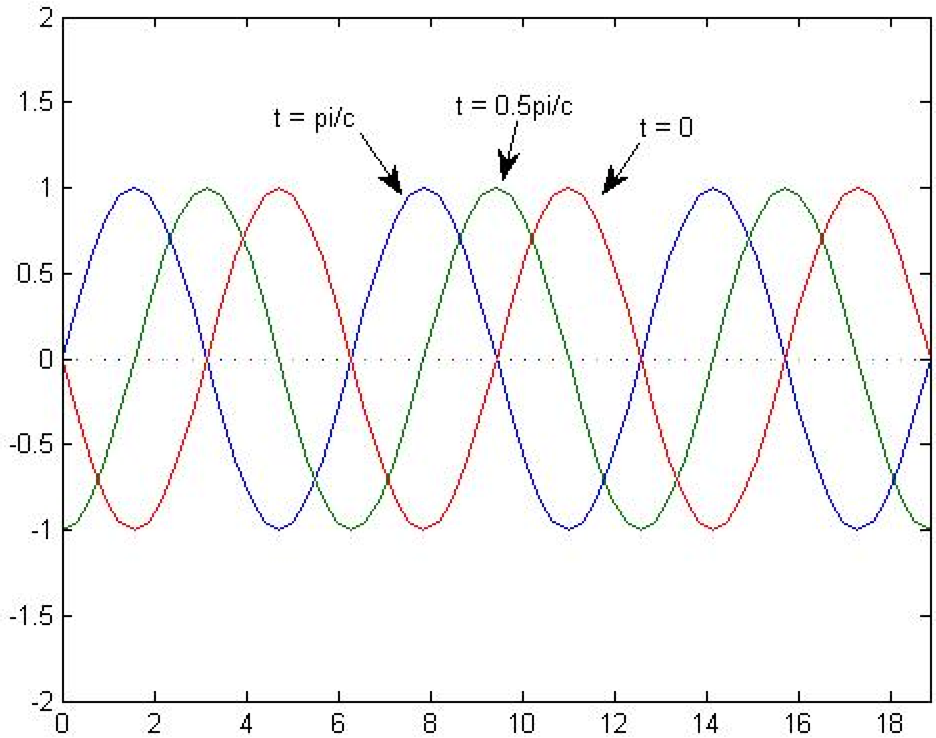
A typical solution: $u(x, t) = F(\xi)$, $\xi \equiv x+ct$; F can be any function that depends only on $x+ct$.
(Exercise: Verify that this is indeed a solution of the original equation.)
The following figure illustrates the behavior of the solution with c = 1. The initial condition, $u(x, t = 0)$, is a "top hat" structure. At later times, this structure moves to the left with a "speed" of $\delta x/\delta t = -1$ while preserving its shape. (The δx and δt here are the increments in space and time in the following diagrams.)



The 3 panels are $u(x, 0)$, $u(x, 1)$, and $u(x, 2)$

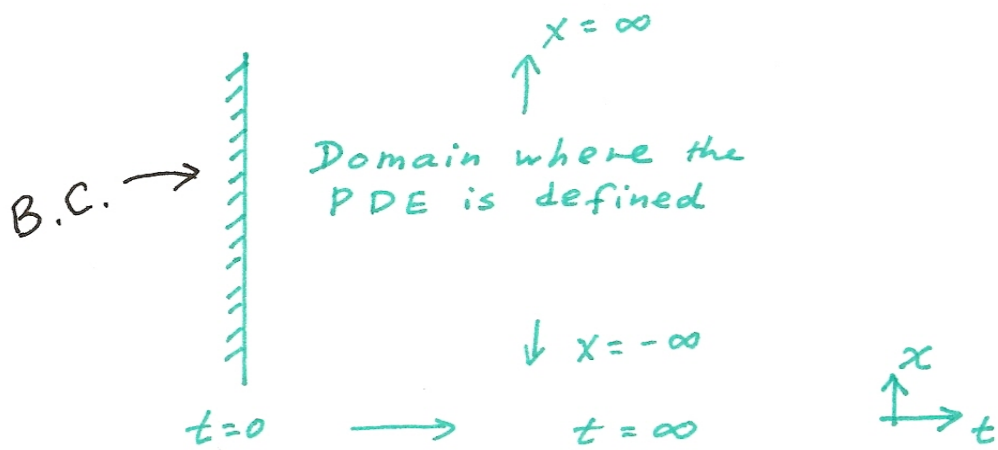
Linear wave equation: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, *describes wave motion*

For example, a simple traveling sinusoidal structure, $u(x, t) = sin(x + ct)$, as illustrated below, is a solution of the equation. (While at this solution is similar to the solution of the linear advection equation, more complicated behavior would emerge when we consider the superposition of different sinusoidal "modes", and when more complicated boundary conditions are introduced for the wave equation.)



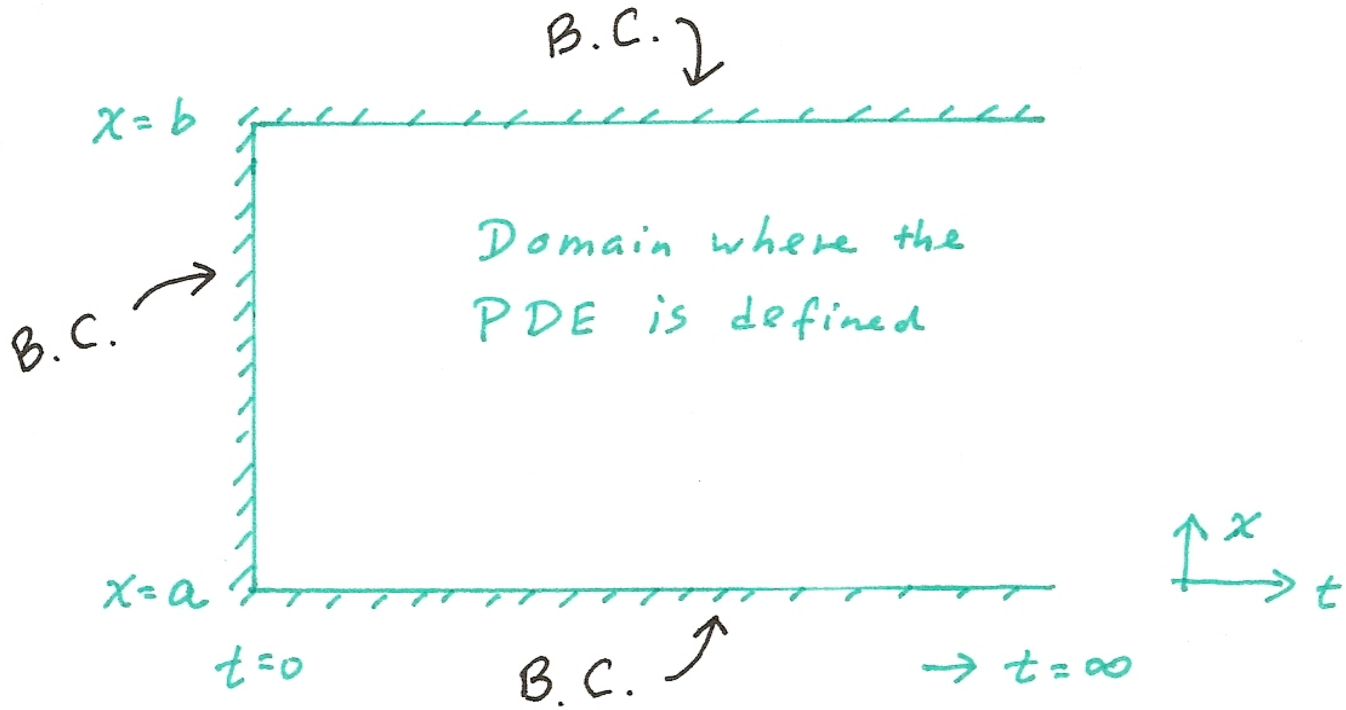
Boundary conditions (I)

In the three preceding examples, we glossed over the role of boundary conditions. The solutions of the heat equation and linear advection equation described before are valid for an unbounded domain in space, $x \in (\infty, -\infty)$, and a "semi-infinite" domain in time, $t \in [0, \infty)$. The first boundary condition we have is simply that u is well-behaved as $x \to \infty$ and $x \to -\infty$. We also need a boundary condition (essentially an "initial condition" in t), $u(x, 0) = G(x)$. The following diagram illustrates the relevant domain in the x-t plane.



Boundary conditions (II)

In real world applications, the heat equation is often defined on a finite interval in x, $x \in [a, b]$, and on a semi-infinite domain in t (consider $u(x,t)$ as the temperature distribution along a finite metal rod at a given time, t). The following diagram illustrates the relevant domain in the x-t plane in this case. In addition to the boundary condition at $t = 0$, $u(x,0) = G(x)$, two more b.c.'s are needed at $x = a$ and $x = b$ for all t. They can be written as $u(a, t) = P(t)$ and $u(b, t) = Q(t)$. Note that $G(x)$ itself has to satisfy the two boundary conditions, $G(a) = P(0)$ and $G(b) = Q(0)$.

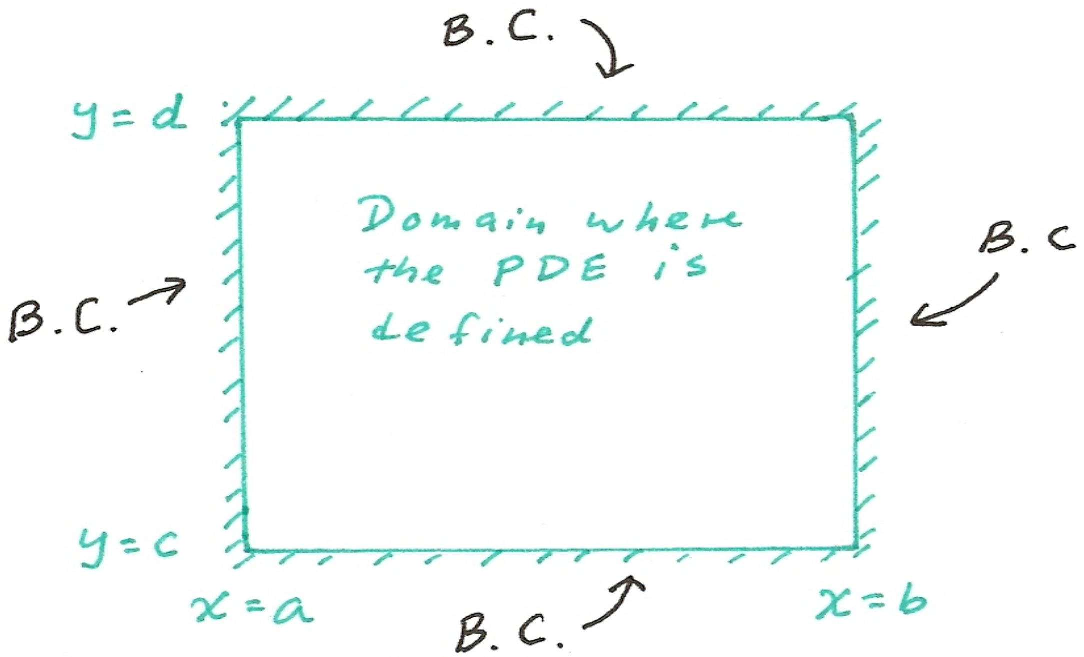


Boundary conditions (III) - Laplace's equation

There are yet other situations when a PDE is defined on a closed domain. A famous example is

Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. (It belongs to the more general class of *elliptic equations*.)

The closed domain is illustrated in the following. In this case, boundary conditions need to be specified at all of the four walls.



Remark: Different types of PDEs often need to be matched with different types of boundary conditions in order for their solutions to exist and be unique.

Heat equation in two- and three-dimensions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \text{(2-D)}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad \text{(3-D)}$$

The behavior of the solutions of these equations is similar to that of the 1-D heat equation. An initially concentrated distribution in u will spread in space as t increases.

For a closed domain with u specified on the "walls", the solution may reach "equilibrium" as $t \to \infty$. At this limit, u ceases to change further so $\partial u/\partial t \approx 0$. Then, the heat equation is reduced to Laplace's equation. In other words, Laplace's equation describes the equiribrium solution (or "steady state solution") of the heat transfer or diffusion problem.

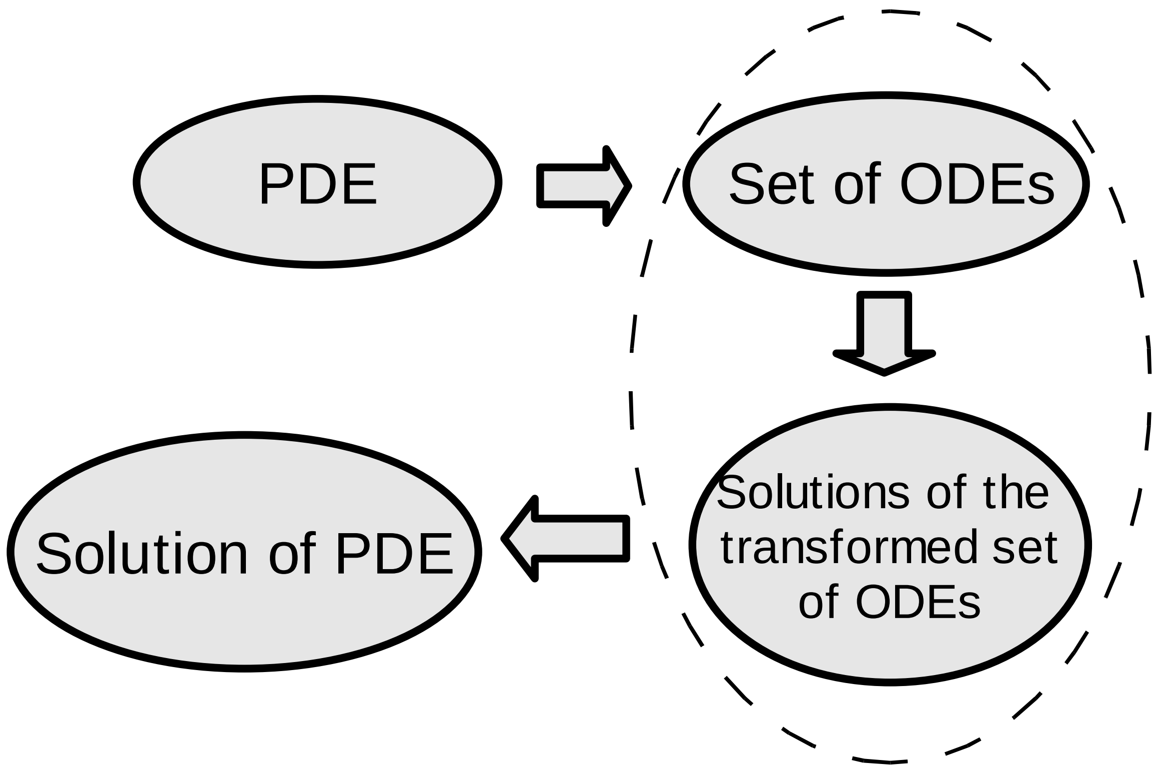
Where to begin?

If you can't solve a problem, then there's an easier problem that you can solve. Find it.

-- G. Polya

Most of the techniques for solving a PDE involves transforming the equation into *something simpler* that we can solve

Example: Method of Fourier series expansion / Spectral method



Example: Finite difference method for Laplace's equation