MAE502 Spring 2011 Homework #1

Problem 1 (5 points) (1 point ≈ 1 % of your total score for the course)

(a) Solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

for $u(x,t)$ defined within the domain of $0 \le x \le 3$ and $t \ge 0$, given the boundary conditions

(i) $u(0, t) = 0$ (ii) $u_x(3, t) = 0$ (Note: $u_x \equiv \partial u/\partial x$)
(iii) $u(x,0) = F(x)$, where $F(x) = 2 \sin(0.5\, \pi\, x) + \sin(2.5\, \pi\, x)$.

Plot the solution as a function of x at $t = 0$ (the "initial state"), 0.01, 0.1, 0.5, and 1.0.

> The above PDE describes the evolution of the temperature, u(x,t), along a thin metal rod with length = 3 that is thermally insulated at its surface except the two end points ($x = 0$ and $x = 3$). The 1st b.c. demands that the temperature at the left end point be fixed to 0, which may be achieved by attaching that end of the rod to a thermal bath with $u = 0$. The 2nd b.c. is equivalent to having the right end of the metal rod thermally insulated, allowing no heat flux to go through it. The third b.c. prescribes the initial distribution of temperature along the metal rod.

(b) What is the equilibrium solution for the system in (a)? Does your full solution in (a) approach the equilibrium solution at a large t ?

(c) At $x = 0$, temperature is fixed but heat flux is allowed to change. Based on your solution in (a), calculate and plot the heat flux, $\phi \equiv -\,\partial u/\partial x$, at $x = 0$ as a function of t. (Note that this plot would track the rate of external heat gain or loss for the whole metal rod due to the heat flow through its left end, as one can readily prove that $dE/dt = \phi(0, t)$, where E is the integral of u over the metal rod.)

(d) The parameter, $S(t) \equiv \sqrt{\int_0^3 (\partial u/\partial x)^2\, dx}$, is a useful measure of the sharpness of temperature contrast within the metal rod; A smoother temperature distribution has a smaller value of $S(t)$. Using your solution for u, evaluate and plot $S(t)$ as a function of t. (For clarity, you might choose to plot $ln[S(t)]$ instead of $S(t)$ if the latter evolves too rapidly with t.)

(e) Using the results from (a)-(d), interpret the behavior of your solution. (If you are paid by a sponsor to solve this heat transfer problem, what story would you be able to tell based on the solution?)

Problem 2 (1 point)
Find the general solution (disregarding boundary conditions), $u(x,y)$, of the following PDE by the method of separation of variables

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + yu = 0$$

Problem 3 (2 points)
(a) Find the solution, $u(x,y)$, of the following PDE + boundary conditions. Here, $u(x,y)$ is defined on the entire 1st quadrant ($x \geq 0$, $y \geq 0$) of the x-y plane (see diagram below) and the two b.c.'s are imposed along the positive x-axis and positive y-axis, i.e., the bottom and left boundaries of the domain.

$$\frac{\partial^2 u}{\partial x \partial y} + u - 2 = 0 \ , \qquad u(x,0) = e^{x} + 2, \quad u(0,y) = e^{-y} + 2,$$

(b) Plot a color/contour map of your solution for the sub-domain of $0 \leq x \leq 1$, $0 \leq y \leq 1$. (See Matlab Example 2 in our course website for useful Matlab commands.)