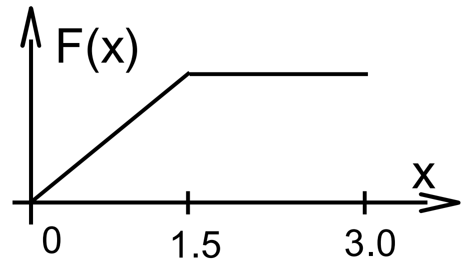
MAE502 Spring 2011 Homework #2

Problem 1 (3 points)

This is a repeat of Prob. 1 in Homework #1 but with a more complicated initial distribution of u. For $u(x,t)$ defined within the domain of $0 \le x \le 3$ and $t \ge 0$, solve the Heat equation



$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$, with boundary conditions

(i) $u(0, t) = 0$
(ii) $u_x(3, t) = 0$ (Note: $u_x \equiv \partial u/\partial x$)
(iii) $u(x,0) = F(x)$, where $F(x) = x$, if $0 \le x \le 1.5$, and $F(x) = 1.5$, if $1.5 < x \le 3$

Plot the solution as a function of x, at $t = 0.1$, 0.8, and 4.0, along with the "initial state" at $t = 0$. It is preferable that all 4 curves are collected in one plot. Discuss the behavior of your solution as $t \to \infty$. (Note: It is part of your job to determine the appropriate number of terms to keep in the infinite series to ensure that the solution is accurate. As a useful measure, the solution at $t = 0$ should match the given initial state, $F(x)$.)

Problem 2 (5 points)

Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$

for $u(x, y)$ defined within the domain of $0 \le x \le 1$ and $0 \le y \le 1$, given the following boundary conditions,

(i) $u(0, y) = 0$
(ii) $u(1, y) = \sin(\pi y) + 0.5 \sin(2\pi y)$
(iii) $u(x, 0) = 0$
(iv) $u(x,1) = 5 \sin(0.5\pi x) - 5x$

Make a color/contour plot of your solution.
What are the values of $u(x, y)$ at (i) $x = 0.4$, $y = 0.3$, and (ii) $x = 0.6$, $y = 0.7$?

Problem 3 (1 point)
For the heat transfer problem, the Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K \frac{\partial^2 u}{\partial \hat{x}^2} \quad , \; 0 \le \hat{x} \le L \quad (L \text{ is the length of the "metal rod", in meters}),$$

where $\hat{t}$ and $\hat{x}$ are time in seconds and distance in meters, K is thermal diffusivity in m^2/s, and u is temperature. In our discussion, we usually consider the non-dimensional form of the equation with u defined on the interval, $0 \le x \le 1$, where x is related to $\hat{x}$ by $\hat{x} = L\,x$. In addition, time is also non-dimensionalized by $\hat{t} = T\,t$, where T is a certain dimensional time scale. We also often ignore K because (as long as K is constant) the transformation, $x' = x/\sqrt{K}$, or $t' = K\,t$, would eliminate the explicit dependence on K. The combination of these transformations lead to a simpler nondimensional problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , \; 0 \le x \le 1 \;\; (\text{and } t \ge \infty) \qquad (1)$$

Nevertheless, once the non-dimensional problem is solved, we need to transform the answer back to the real world situation in order for the solution to be useful.

(a) Suppose that the nondimensionalized heat equation, Eq. (1), is used to model the real world situation of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem?

(b) Same as (a), but suppose that Eq. (1) describes heat transfer along a wooden stick that is 0.3 meter long and made of pine wood ($K \approx 10^{-7}$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to? (We consider $t = 0.01$ because it is about the time when a significant redistribution of temperature begins to take place in the scenario described in Part (c).)

(c) Are the time scales you obtained in (a) and (b) consistent with daily experience? For instance, one can use a long wooden spoon to continuously stir a boiling pot of soup without getting one's hand burned. In contrast, the same practice would make one very uncomfortable if the spoon is made entirely of copper. Note that the time scale for cooking a pot of soup is about 10 minutes. The length of a big wooden spoon is about a foot, or 0.3 meter.