MAE 502 Spring 2011 Homework #4

Prob 1 (2 points)
An engineering problem is defined on the interval of $1 \le x \le 2$ and for $t \ge 0$. It is governed by the following PDE and boundary conditions,

$$e^{-2x} \frac{\partial u}{\partial t} = 0.5 \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + x^2 e^{-2x} u \quad ,$$

$$u(1, t) = 0 \quad , u(2, t) = 0, \quad u(x,0) = P(x),$$

where $P(x)$ is a well-behaved function. (The detail of $P(x)$ is not critical here.) Suppose that you are given a software that solves the standard Sturm-Liouville eigenvalue problem. Describe your strategy to solve the PDE assisted by that software. (Sketch the procedure for an end-to-end solution.)

Prob 2 (2 points)
(a) Given the following function defined on the semi-infinite interval, $0 \le x < \infty$,

$$f(x) = 1 \quad , 0 \le x \le 1, \qquad \text{Eq. (1)}$$
$$\quad = 0 \quad , 1 < x \quad ,$$

determine the Fourier Sine transform of $f(x)$, $F(\omega)$, that satisfies

$$f(x) = \int_0^\infty F(\omega) \sin(\omega x) d\omega \quad .$$

Plot $F(\omega)$ as a function of ω for the range $0 \le \omega \le 30$.

(b) If the $f(x)$ in Eq. (1) is instead defined on a finite interval, $0 \le x \le L$ (but otherwise retains its definition in Eq. (1); we now have $f(x) = 0$ for $1 < x \le L$), find the coefficients, a_n, for the Fourier Sine series of $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n \pi x}{L}\right) \quad .$$

Plot a_n as a function of n for the following cases: (i) For $L = 2$, plot a_n for the range $1 \le n < 60/\pi$. (ii) For $L = 5$, plot a_n for $1 \le n < 150/\pi$. (iii) For $L = 100$, plot a_n for $1 \le n < 3000/\pi$. Compare these plots with the plot of $F(\omega)$ in (a). Discuss your results. (*Note: This homework illustrates the correspondence between Fourier series and Fourier integral.*)

Prob. 3 (5 points)
(a) Using the Fourier transform method, solve the slightly modified heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - bu \quad ,$$

defined on the infinite interval, $-\infty < x < \infty$, given the boundary conditions

(I) $u(x, t)$ and all of its partial derivatives in x vanish as $x \to \infty$ and $x \to -\infty$.

(II) $u(x,0) = \mathrm{P}(x)$, where

$$\begin{aligned} \mathrm{P}(x) &= 0.5 \,, \quad 1 \le x \le 3 \\ &= 1 \,, \quad -2 \le x \le -1 \\ &= 0 \,, \quad \text{otherwise} . \end{aligned}$$

(b) Using your solution, for $b = 0$, evaluate and plot $u(x, t)$ as a function of x at $t = 0.1$, 0.3, and 1. As a reference, please also plot the initial state $u(x,0) = \mathrm{P}(x)$. In addition, evaluate and plot $u(x, t)$ at $t = 1$ for the case with $b = 1$. Discuss your results.
Note: Part (b) is important and accounts for at least 50% of the score. Numerical integration (e.g., by the trapezoidal method) may be needed to evaluate $u(x, t)$. Since numerical integration cannot go all the way to ∞, one has to "truncate" the integral at a finite value of ω. This is analogous to truncating a Fourier series at a finite wavenumber n. A useful way to determine where to truncate the integral is to plot, for a give t, $U(\omega, t)$ (the Fourier transform of $u(x, t)$) as a function of ω and observe how $U(\omega, t)$ decays with ω.

(c) Define $E(t) \equiv \int_{-\infty}^{\infty} u(x,t)\,dx$, show that

$$\frac{dE}{dt} = -bE \quad .$$

In particular, when $b = 0$, E is "conserved", i.e., it does not change with t. Does your solution in Part (a) satisfy this conservative property?