Solving nonhomogeneous PDEs by Fourier transform

Example: For $u(x, t)$ defines on $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t) \ , \qquad (1)$$

with boundary conditions

(I) $u(x, t)$ and its partial derivatives in x vanishes as $x \to \infty$ and $x \to -\infty$

(II) $u(x,0) = \mathrm{P}(x)$

Recall Fourier transform pair

Fourier transform of u: $\mathcal{F}[u] \equiv U(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{i\omega x}dx \qquad (2)$

Inverse transform: $\mathcal{F}^{-1}[U] \equiv u(x,t) = \int_{-\infty}^{\infty} U(\omega,t)\mathrm{e}^{-i\omega x}d\omega \qquad (3)$

From Eq. (2) we have

$$\mathcal{F}[u(x,0)] \equiv U(\omega,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(x)\mathrm{e}^{i\omega x}dx \qquad (4)$$

Solution:
Take Fourier transform of the PDE

$$\mathcal{F}\left[\ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t)\ \right]$$

$$\Rightarrow \quad \frac{\partial U}{\partial t} = -\omega^2 U + Q(\omega, t) \ , \qquad (5)$$

where

$$Q(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(x,t)\,\mathrm{e}^{i\omega x}\,dx \ . \qquad (6)$$

Note: We use Fourier transform because the transformed equation in "Fourier space", or "spectral space", Eq. (5), is much simpler than the original PDE in physical space, Eq. (1)

The solution of Eq. (4) is (cf. Sec 8.3)

$$U(\omega,t) = U(\omega,0)\,\mathrm{e}^{-\omega^2 t} + \int_0^t Q(\omega,t')\,\mathrm{e}^{-\omega^2(t-t')}\,dt' \ . \qquad (7)$$

Procedure: Evaluate $U(\omega,0)$ from Eq. (4); Evaluate $Q(\omega, t)$ from Eq. (6); Evaluate $U(\omega,t)$ from Eq. (7); Evaluate $u(x, t)$ (final answer) from Eq. (3)