Separation of variables

Idea: Transform a PDE of 2 variables into a pair of ODEs (more precisely, 2 families of ODEs)

Example 1: Find the general solution of $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$

Step 1. Assume that $u(x,y) = G(x)H(y)$, i.e., u can be written as the product of two functions, one depends only on x, the other depends only on y. This leads to

$$H\frac{dG}{dx} + G\frac{dH}{dy} = 0 \quad . \qquad (1)$$

Step 2. Rearrange the equation to collect all that depend only on x in one side, and all that depend only on y in the other. This can be done by multiplying Eq. (1) by 1/(GH),

$$\Rightarrow \quad \frac{1}{G}\frac{dG}{dx} = \frac{-1}{H}\frac{dH}{dy} \quad . \qquad (2)$$

Step 3. Since the l.h.s. of Eq. (2) depends only on x and r.h.s depends only on y, the only way for the equation to hold is to have l.h.s = r.h.s = *a common constant*. ("common" is the key word here.) Thus, Eq. (2) implies

$$\frac{1}{G}\frac{dG}{dx} = \frac{-1}{H}\frac{dH}{dy} = c \quad ,$$

where c is a constant.

(continued)

The original PDE is now split into two ODEs,

$$\frac{1}{G}\frac{dG}{dx} = c \quad , \tag{3}$$

$$\frac{1}{H}\frac{dH}{dy} = -c \quad . \tag{4}$$

The general solutions for (3) and (4) are $G(x) = k_1 \exp(cx)$ and $H(y) = k_2 \exp(-cy)$, where k_1 and k_2 are just two arbitrary constants that can be combined later.

Step 4. Multiply G to H to reconstruct u :

$$u(x, y) = k \exp(cx - cy) \quad ,$$

where k is an arbitrary constant. It can be readily verified that this solution satisfies the original PDE.

(Although the procedure here looks simple, more complicated situations will emerge once boundary conditions are introduced. This will be discussed shortly.)

Example 2: Find the general solution of $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0$

Again, assume u(x, y) = G(x)H(y) so the PDE is transformed into

$$y\,H\frac{d\,G}{d\,x} - x\,G\frac{d\,H}{d\,y} = 0 \quad .$$

Following the argument in Ex. 1, the above equation implies

$$\frac{1}{x\,G}\frac{d\,G}{d\,x} = \frac{1}{y\,H}\frac{d\,H}{d\,y} = c \quad .$$

The PDE is transformed into 2 ODEs,

$$\frac{1}{x\,G}\frac{d\,G}{d\,x} = c \quad , \tag{5}$$

$$\frac{1}{y\,H}\frac{d\,H}{d\,y} = c \quad . \tag{6}$$

From (5) and (6) we obtain G(x) = k_1 exp(c $x^2/2$) and H(y) = k_2 exp(c $y^2/2$). Recombine them, we have

$$u(x, y) = k \exp[(c/2)\,(x^2 + y^2)] \quad .$$

Quick note: In the preceding example, how do we solve the ODE, $\frac{1}{xG}\frac{dG}{dx} = c$?

(1) $x\,dx = (1/2)\,d(x^2)$; change of variable $z \equiv x^2$ ==> $x\,dx = (dz)/2$

(2) $dG/G = d(\ln G)$; change of variable $Y \equiv \ln G$ ==> $dG/G = dY$

From (1) and (2), the ODE becomes $\frac{dY}{dz} = c/2$ ==> $Y = c\,z/2 + k$, where k is a constant. So, the solution is

$$\ln G = c\,x^2/2 + k \quad ==> \quad G(x) = k\exp(c\,x^2/2)$$

Exercise: Find the general solutions of the following ODEs

(i) $\frac{1}{x^2}\frac{dG}{dx} = c$ (ii) $x\frac{dG}{dx} = c$ (iii) $x^2\frac{dG}{dx} = c$

(iv) $x^2\frac{d^2G}{dx^2} - 2x\frac{dG}{dx} + 2G = 0$ (v) $\frac{d^2G}{dx^2} - 5\frac{dG}{dx} + 4G = 0$

Heat equation - An example of end-to-end solution

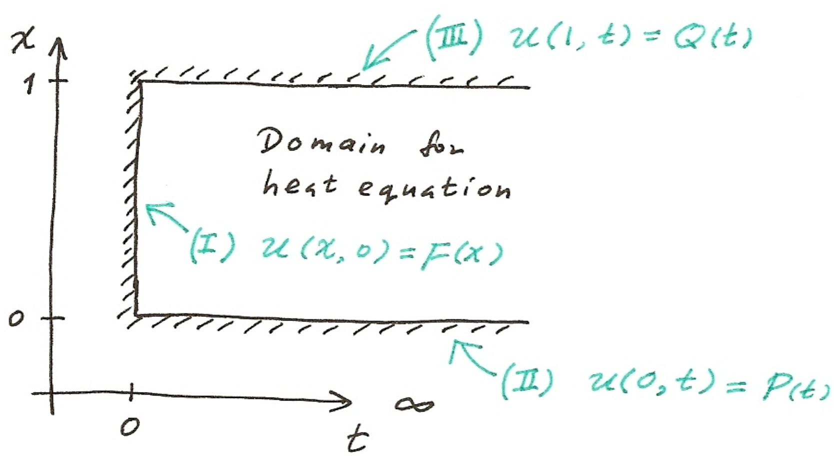
When boundary conditions are considered, the method of separation of variables usually leads to an eigenvalue problem

Example 3: For $u(x, t)$ defined on $x \in [0, 1]$ and $t \in [0, \infty)$, solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with boundary conditions ((III) describes the "initial state" of u) :
(I) $u(0, t) = 0$, (II) $u(1, t) = 0$, (III) $u(x, 0) = 4\sin(3\pi x) + 7\sin(8\pi x)$

This is the example given in **pp. 38-48 in textbook**. In this case, the heat equation is defined on the semi-open domain in t-x plane bounded by $t = 0$, $x = 0$, and $x = 1$. See Slides #1. The relevant diagram from Slides #1 is repeated here, with $F(x) = 4\sin(3\pi x)+7\sin(8\pi x)$, $P(t) = 0$, and $Q(t) = 0$.



Step 1: Separation of variables. Let $u(x, t) = \mathrm{G}(x)\mathrm{H}(t)$, the usual procedure leads to

$$\frac{1}{G}\frac{d^2 G}{d x^2} = \frac{1}{H}\frac{d H}{d y} = c \quad .$$

So, the PDE is converted to two ODEs,

$$\frac{1}{G}\frac{d^2 G}{d x^2} = c \quad , \tag{7}$$

$$\frac{1}{H}\frac{d H}{d y} = c \quad . \tag{8}$$

Since boundary conditions (I) and (II) have to be satisfied for all t, they are reduced to

(IV) $\mathrm{G}(0) = 0$, (V) $\mathrm{G}(1) = 0$,

for Eq. (7).

We will see that in order for the solution of Eq. (7) to satisfy (IV) and (V) and be non-trivial (i.e., $\mathrm{G}(x)$ is not identically zero), the "constant" c must be a certain specific values. **Equation (7) plus the boundary conditions (IV) and (V) form an eigenvalue problem**.

Step 2: Solve the eigenvalue problem, Eq. (7) + b.c. (IV) and (V), to obtain the eigenvalue c and eigenfunction G(x).

It can be readily shown that the case with $c \geq 0$ leads to trivial solution, $G(x) \equiv 0$.

When $c < 0$, write $c = -k^2$; The general solution of Eq. (7) is

$$G(x) = C \sin(k\,x) + D \cos(k\,x)\ .$$

From boundary condition (IV), $D = 0$. From b. c. (V), and demanding that $G(x)$ be non-trivial, we have

$$\sin(k) = 0\ .$$

The only values of k that satisfy this condition are $k_1 = \pi$, $k_2 = 2\pi$, ..., $k_N = N\pi$, ... Thus, our eigenvalues are

$$c_1 = -\pi^2,\ \ c_2 = -4\,\pi^2,\ \ c_3 = -9\,\pi^2\ , \ldots\ c_N = -N^2\,\pi^2\ ,\ldots \tag{9}$$

and the corresponding eigenfunctions are

$$G_1(x) = \sin(\pi x),\ \ G_2(x) = \sin(2\pi x),\ G_3(x) = \sin(3\pi x),\ \ldots\ G_N(x) = \sin(N\pi x),\ \ldots \tag{10}$$

Step 3: For a given eigenvalue c = c_N, solve the other ODE for H(t). This leads to

$$H_N(t) = exp(c_N t) = \exp(-N^2 \pi^2 t) \qquad (11)$$

(Again, remember that the "c" in Eqs. (7) and (8) is a common constant for both equations.)

Step 4: Combine $G_N(x)$ and $H_N(t)$ to form the full eigenfunction,

$$u_N(x, t) = G_N(x)\, H_N(t) = sin(N \pi x) \exp(-N^2 \pi^2 t) \,. \qquad (12)$$

Here, the subscript "N" indicates the N-th eigenfunction.

Step 5: Represent the full solution as the linear combination of all eigenfunctions.

Since each of the eigenfunctions satisfies the PDE+b.c. (IV) & (V), and since the PDE is linear, any linear combination of the eigenfunctions is also a solution to the PDE+ b.c.'s. Then, the most general form of the solution is

$$\begin{aligned} u(x, t) &= a_1\, u_1(x, t) + a_2\, u_2(x, t) + a_3\, u_3(x, t) + \ldots \\ &= \sum_{n=1}^{\infty} a_n\, u_n(x, t) \end{aligned} \qquad (13)$$

Here, the coefficients, a_n, are yet to be determined.

Step 6: Use the b. c. (III) at $t = 0$ to determine the coefficients, a_n, in Eq. (12).

From Eqs. (12) and (13) we have, at $t = 0$,

$$u(x,0) = \sum_{n=1}^{\infty} a_n\, u_n(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad . \qquad (14)$$

In order for this to satisfy b.c. (III), $u(x, 0) = 4\sin(3\pi x) + 7\sin(8\pi x)$, we must have:

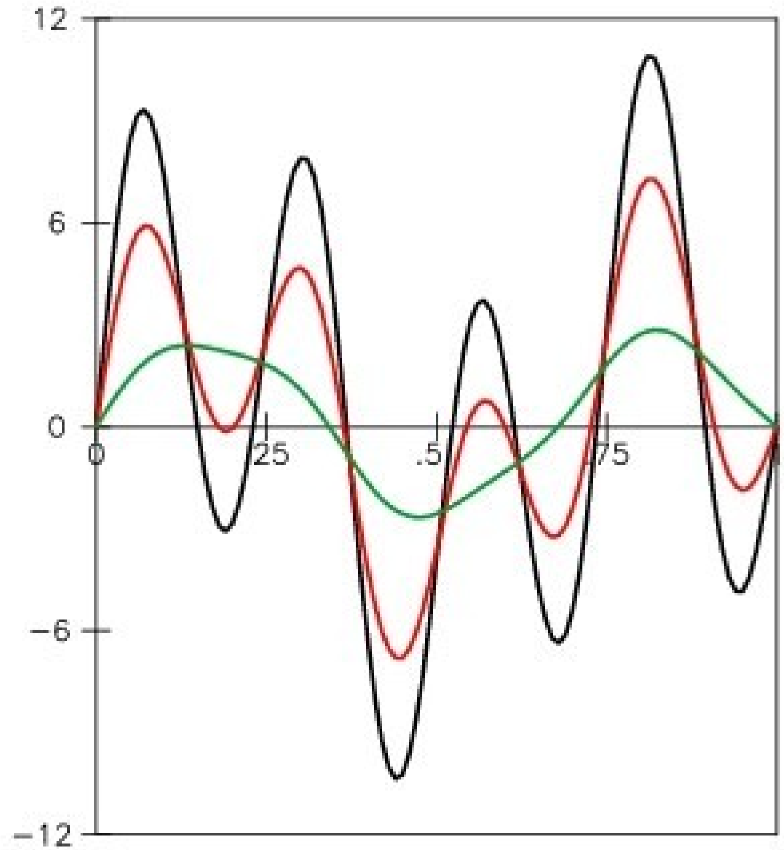
$a_3 = 4$, $a_8 = 7$, and $a_n = 0$ for all other n .

(Important: We are luck that the initial condition is very simple. In general, we need to use an orthogonality relation to determine the expansion coefficient.)

Using this last piece of information and Eqs. (12) and (13), we obtain the complete solution as

$$u(x, t) = 4\sin(3\pi x)\exp(-9\pi^2 t) + 7\sin(8\pi x)\exp(-64\pi^2 t) .$$

It satisfies the PDE and all three boundary conditions. The figure in next page is a plot for the solution $u(x, t)$ at $t = 0$, $t = 0.001$, and $t = 0.005$. **Note that the temperature** distribution, u, becomes smoother over time. This is an important property of the solution of the heat (or "diffusion") equation.



The solution, $u(x,t)$ at $t = 0$ (black), 0.001 (red), and 0.005 (green)