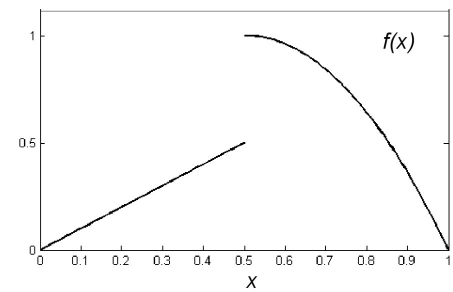
MAE/MSE 502 Spring 2011 Homework #3

Prob. 1 (3 points)

(a) Given the function defined on the interval, $0 \le x \le 1$,



$$f(x) = x, \quad 0 \le x \le 1/2$$
$$= 4(x - x^2), \quad 1/2 < x \le 1 \quad ,$$

work out the Fourier Sine series expansion,

$$F_S(x) \approx \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad ,$$

where $F_S(x)$ denotes the Fourier Sine series representation of $f(x)$. A sketch of $f(x)$ is shown at right; Notice a discontinuity at $x = 1/2$.

(b) Plot the original $f(x)$ and its Fourier Sine series representation, $F_S(x)$, truncated (inclusively) at $n = 5$, 10, and 30. Please collect all four curves in a single plot. What are the values of $F_S(x)$ at $x = 0.75$ for the three cases truncated at $n = 5$, 10, and 30? Compare them to the exact value, $f(0.75)$, to determine the percentage error (using the exact value as denominator) for the three cases. Repeat the exercise for $x = 0.51$ (a point close to the discontinuity). Discuss the results.

(c) Find the analytic expression (as an infinite series) for the value of $F_S(x)$ at $x = 1/2$. Define $S(\mathrm{N})$ as the value of $F_S(1/2)$ calculated from the Fourier Sine series truncated at $n = \mathrm{N}$, plot $S(\mathrm{N})$ as a function of N for the range of $1 \le \mathrm{N} \le 30$. What value does $S(\mathrm{N})$ converge to at large N?

Prob. 2 (4 points)

(a) Solve the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad ,$$

for $u(x, y, t)$ defined within the domain of $0 \le x \le 1$, $0 \le y \le 1$, and $t \ge 0$, with the following boundary conditions

(i) $u(0, y, t) = 0$
(ii) $u(1, y, t) = 0$
(iii) $u(x, 0, t) = 0$
(iv) $u(x, 1, t) = 0$
(v) $u(x, y, 0) = [x + \cos(0.5\pi x) - 1][\sin(\pi y)+\sin(2\pi y)]$
(vi) $u_t(x, y, 0) = 0 \quad (u_t \equiv \partial u/\partial t)$.

(b) Plot the solution $u(x, y, t)$ at $t = 0$ (initial state), 0.3, 0.7, and 2.47 as contour/color maps.

Prob. 3 (2 points)
To prepare for the upcoming discussion for Chapter 5, let us consider the following eigenvalue problem,

$$\frac{d^2G}{dx^2} = c\ G \quad , \quad G'(0) = 1 \ , \ G(1) = 0 \quad (G' \text{ is } dG/dx) \ .$$

(a) Determine the eigenvalues and the corresponding eigenfunctions of this problem.
Do consider all three possibilities with $c > 0$, $c = 0$, and $c < 0$. Are the eigenvalues discrete? For instance, if the boundary conditions are replaced by the familiar $G(0) = 0$ and $G(1) = 0$, we would have $c = c_n = -n^2 \pi^2$ (n is an integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation when the eigenvalues are not discrete is if all values within an interval, $A < c < B$, are valid eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

(b) Plot the eigenfunctions, $G_C(x)$, associated with the eigenvalues $c = -1, -2, -3, 0, 1, 10$, and 100. Collect them in a single figure. You will find in Part (a) that all of these values of c are indeed valid eigenvalues.

(c) Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^1 G_p(x)\, G_q(x)\, dx = 0, \quad if\ p \neq q \quad ,$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions that correspond to two distinctive eigenvalues p and q? Please provide a concrete discussion to support your conclusion. Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of $G_p(x)G_q(x)$ and show that it leads to a non-zero value even when $p \neq q$. One such counterexample would suffice to prove that the orthogonality relation does not stand. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all p and q.