**MAE/MSE 502 Spring 2012 Homework #4**

**Prob 1 (1 point)**

An engineering problem is defined on the interval of $1 \le x \le 2$ and for $t \ge 0$. It is governed by the following PDE and boundary conditions,

$$\frac{\partial u}{\partial t} + 3x^2 \frac{\partial^2 u}{\partial x^2} + 6x \frac{\partial u}{\partial x} + e^{-x} u = 0 \quad ,$$

$$u(1, t) = 0 \ , \ u(2, t) = 0, \ u(x,0) = P(x),$$

where $P(x)$ is a well-behaved function. (The detail of $P(x)$ is not critical here.) Suppose that you are given a software that solves the standard Sturm-Liouville eigenvalue problem. Describe your strategy to solve the PDE assisted by that software. (<u>Sketch the procedure for an end-to-end solution.</u>)

**Prob 2 (1 point)**

**(a)** Given the following function defined on the semi-infinite interval, $0 \le x < \infty$,

$$f(x) = 1 \ , \ 0 \le x \le 1, \qquad \text{Eq. (1)}$$
$$\quad = 0 \ , \ 1 < x \ ,$$

determine the Fourier Sine transform of $f(x)$, $F(\omega)$, that satisfies

$$f(x) = \int_0^\infty F(\omega) \sin(\omega x)\, d\omega \ .$$

Plot $F(\omega)$ as a function of $\omega$ for the range $0 \le \omega \le 30$.

**(b)** If the $f(x)$ in Eq. (1) is instead defined on a finite interval, $0 \le x \le L$ (but otherwise retains its definition in Eq. (1); we now have $f(x) = 0$ for $1 < x \le L$), find the coefficients, $a_n$, for the Fourier Sine series of $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \ .$$

Plot $a_n$ as a function of $n$ for the following cases: (i) For $L = 2$, plot $a_n$ for the range $1 \le n < 60/\pi$. (ii) For $L = 5$, plot $a_n$ for $1 \le n < 150/\pi$. (iii) For $L = 100$, plot $a_n$ for $1 \le n < 3000/\pi$. Compare these plots with the plot of $F(\omega)$ in (a). <u>Discuss your results</u>. (*Note: This homework illustrates the correspondence between Fourier series and Fourier integral*.)

**Prob. 3 (5 points)**

(a) Using the Fourier transform method, solve the modified Heat equation defined on the infinite interval, $-\infty < x < \infty$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 0.01\,\frac{\partial^4 u}{\partial x^4} \ ,$$

with the given the boundary conditions:

(I) $u(x, t)$ and all of its partial derivatives in $x$ ($u_x$, $u_{xx}$, etc.) vanish as $x \to \infty$ and $x \to -\infty$.

(II) $u(x,0) = \mathrm{P}(x)$ , where

$$\begin{aligned} \mathrm{P}(x) &= 1 + x\ , && -1 \le x \le 0 \\ &= 1 - x, && 0 < x \le 1 \\ &= 0.5\ , && 2 \le x \le 3 \\ &= 0\ \ , && \text{otherwise.} \end{aligned}$$

(b) Using your solution, evaluate and plot $u(x, t)$ as a function of $x$ at $t = 0.1$, 0.3, and 1. As a reference, please also plot the initial state $u(x,0) = \mathrm{P}(x)$ and collect all 4 curves in one figure. Discuss your results.

*Note: Part (b) is important and accounts for at least 50% of the score. Numerical integration (e.g., by the trapezoidal method) may be needed to evaluate $u(x, t)$. Since numerical integration cannot go all the way to $\infty$, one has to "truncate" the integral at a finite value of $\omega$. This is analogous to truncating a Fourier series at a finite wavenumber $n$. A useful way to determine where to truncate the integral is to plot, for a give $t$, $U(\omega, t)$ (the Fourier transform of $u(x, t)$) as a function of $\omega$ and observe how $U(\omega, t)$ decays with $\omega$.*

(c) Define $E(t) \equiv \int_{-\infty}^{\infty} u(x,t)\,dx$ , show that

$$\frac{dE}{dt} = 0 \ .$$

In other words, $E$ is "conserved" (it does not change with $t$).