MAE502 Spring 2013 Homework #1

1 point ≈ 1% of your total score for this class

Problem 1 (3 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with boundary conditions,

(i) $u(0, t) = 3$, (ii) $u(1, t) = 1$, and (iii) $u(x,0) = \mathrm{P}(x)$,

where

$$\begin{aligned} \mathrm{P}(x) &\equiv 3 - 4x , \text{ if } 0 \le x \le 0.5 \\ &\equiv 1 \quad , \text{ if } 0.5 < x \le 1 . \end{aligned}$$

(See Fig. 1 for a plot of P(x).)

Plot the solution as a function of x, at $t = 0.02$, 0.06, and 0.12, along with the initial condition (i.e., P(x)) and the steady state solution ($u(x,t)$ as $t \to \infty$). Please collect all five curves in a single plot. (Minimum 50% deduction without the plot, even if your solution is correct.)

What is the value of $u(x,t)$ at $x = 0.6$, $t = 0.2$?

Note: If your solution is expressed as an infinite series, it is your job to determine the appropriate number of terms to keep in the series to ensure that the solution is accurate. As a useful measure, the solution at $t = 0$ should nearly match the given initial state, P(x), in the 3rd boundary condition. If they do not match, either the solution is wrong or you have not retained enough terms in the series.

Problem 2 (4 points)

Consider the system described in Prob 1 except that the second boundary condition is changed to

(ii) $u_x(1, t) = 0$ (u_x is $\partial u/\partial x$).

Find the solution of this new system and plot the solution as a function of x, at $t = 0.05$, 0.2, 0.5, and 1.0, along with the initial state and the steady state. Please collect all 6 curves in a single plot.

Problem 3 (0.5 point)

Find the <u>stead state solution</u> of the following system defined on the domain of $-1 \le x \le 1$ and $t \ge 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with boundary conditions: (i) $u_x(-1, t) = 0$, (ii) $u_x(1, t) = 0$, and (iii) $u(x,0) = \mathrm{P}(x)$, where

$$\begin{aligned} \mathrm{P}(x) &\equiv 1 , \quad \text{if } -1 \le x \le -0.5 \\ &\equiv 4x^2 , \text{ if } -0.5 < x \le 0.5 \\ &\equiv 1 , \quad \text{if } 0.5 < x \le 1 \end{aligned}$$

(See Fig. 2 for a plot of P(x).)

Problem 4 (0.5 point)

For the heat transfer problem, the Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K \frac{\partial^2 u}{\partial \hat{x}^2} \;, \; 0 \le \hat{x} \le L \;\; (L \text{ is the length of the "metal rod", in meters) and } \hat{t} \ge 0 \,, \qquad (1)$$

where $\hat{t}$ and $\hat{x}$ are time in seconds and distance in meters, K is thermal diffusivity in m^2/s, and u is temperature. In our class, we usually consider the non-dimensionalized version of (1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;, \; 0 \le x \le 1 \text{ and } t \ge 0 \;, \qquad (2)$$

where x is related to $\hat{x}$ by $\hat{x} = L\,x$. In (2), time is also non-dimensionalized by $\hat{t} = T\,t$, where T is a certain dimensional time scale, and so on.

(a) In order to claim that the non-dimensionalized system, (1), is equivalent to its dimensional counterpart, (2), the three parameters K, L, and T must satisfy a unique relation. First, find out what this relation is. (For the discussion in part (b) & (c), it is useful to write the relation as $T = f(K, L)$.)

(b) Suppose that the non-dimensionalized Heat equation, (2), is used to model the real world problem of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem?

(c) Same as (b), but suppose that (2) describes heat transfer along a wooden stick that is 0.3 meter long and made of pine wood ($K \approx 10^{-7}$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to? (We consider $t = 0.01$ because it is about the time when a significant redistribution of temperature begins to take place in the scenario described in Part (d).)

(d) Are the time scales you obtained in (b) and (c) consistent with daily experience? For instance, one can use a long wooden spoon to continuously stir a boiling pot of soup without getting one's hand burned. In contrast, the same practice would make one very uncomfortable if the spoon is made entirely of copper. Note that the time scale for cooking a pot of soup is about 10 minutes. The length of a big wooden spoon is about a foot, or 0.3 meter.

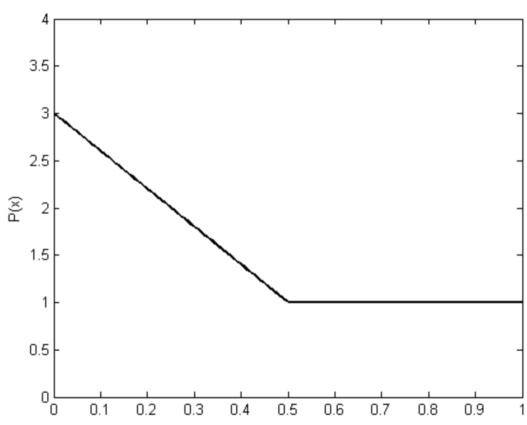

Fig. 1 (for Prob 1 & 2)

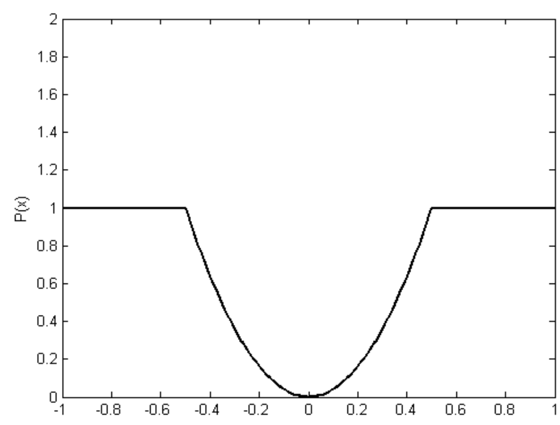

Fig. 2 (for Prob 3)