MAE502 Spring 2013 Homework #3

Prob. 1 (4 points)
Consider the function (see sketch below) defined on the interval of $0 \le x \le 1$,

$$f(x) = 0.5, \qquad 0 \le x \le 0.5$$
$$= 2 - 2x, \quad 0.5 < x \le 1 \ .$$

(a) Work out the Fourier Sine series expansion,

$$F_S(x) \approx \sum_{n=1}^{\infty} a_n \sin(n\pi x) \ ,$$

where $F_S(x)$ denotes the Fourier Sine series representation of $f(x)$. Plot the original $f(x)$ and its Fourier Sine series representation, $F_S(x)$, truncated (inclusively) at $n = 5$, 10, and 30. Please collect all four curves in a single plot.

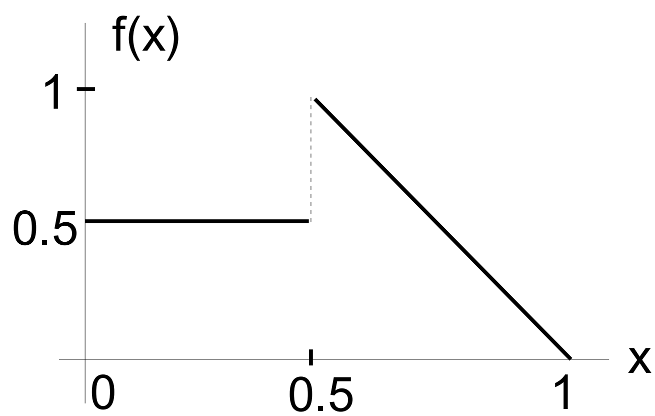
(b) What are the values of $F_S(x)$ at $x = 0.75$ for the three cases truncated at $n = 5$, 10, and 30? Compare them to the exact value, $f(0.75)$, to determine the percentage error (using the exact value as denominator) for the three cases. Repeat the exercise for $x = 0.51$ (a point close to the discontinuity). Discuss the results.

(c) Define $S(\mathrm{N})$ as the value of $F_S(0.5)$ calculated from the Fourier Sine series truncated at $n = \mathrm{N}$, plot $S(\mathrm{N})$ as a function of N for the range of $1 \le \mathrm{N} \le 30$. What value does $S(\mathrm{N})$ converge to at large N?

(d) Repeat (a) but now work out the Fourier Cosine series expansion,

$$F_C(x) \approx \sum_{n=0}^{\infty} a_n \cos(n\pi x) \ ,$$

where $F_C(x)$ denotes the Fourier Cosine series representation of $f(x)$. (Beware that the summation at right starts at $n = 0$.) Plot the $F_C(x)$ truncated (inclusively) at $n = 5$, 10, and 30, along with the original $f(x)$.



Prob. 2 (1.5 points)
Consider the following eigenvalue problem,

$$\frac{d^2 G}{d x^2} = c\, G \ , \quad G(0) = 0 \, , \ G'(1) = 1 \quad (G' \text{ is } dG/dx) \, .$$

(a) Determine the eigenvalues and the corresponding eigenfunctions of this problem. Do consider all three possibilities with c > 0, c = 0, and c < 0. Are the eigenvalues discrete? For instance, if the boundary conditions are replaced by the familiar $G(0) = 0$ and $G(1) = 0$, we would have $c = c_n = -n^2 \pi^2$ (n is an integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation when the eigenvalues are not discrete is if all values within an interval, A < c < B, are valid eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

(b) Plot the eigenfunctions, $G_C(x)$, associated with the eigenvalues c = −10, −5, −1, 0, 1, 5, and 10. Collect them in a single figure. You will find in Part (a) that all of these values of c are indeed valid eigenvalues.

(c) Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^1 G_p(x) G_q(x) dx = 0, \quad \text{if } p \neq q \quad ,$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions that correspond to two distinctive eigenvalues p and q? Please provide a concrete discussion to support your conclusion. Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of $G_p(x)G_q(x)$ and show that it leads to a non-zero value even when $p \neq q$. One such counterexample would suffice to prove that the orthogonality relation does not hold. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all p and q.

Prob. 3 (0.5 point)
(a) Consider the orthogonal basis in the 3-D space: $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1/2, 1/2)$, and $\mathbf{e}_3 = (0, -1/2, 1/2)$. If a given vector, $\mathbf{X} = (3, 5, 4)$, is represented as $\mathbf{X} = a_1\, \mathbf{e}_1 + a_2\, \mathbf{e}_2 + a_3\, \mathbf{e}_3$, find the expansion coefficients a_1, a_2, and a_3.
(b) Consider, instead, a non-orthogonal basis of $\mathbf{d}_1 = (1,1,1)$, $\mathbf{d}_2 = (1, 1, 2)$, and $\mathbf{d}_3 = (1, 0, -2)$. If the $\mathbf{X}$ vector given in (a) is represented by $\mathbf{X} = b_1\, \mathbf{d}_1 + b_2\, \mathbf{d}_2 + b_3\, \mathbf{d}_3$, find the expansion coefficients b_1, b_2, and b_3. Comment on the labor that is required to obtain the expansion coefficients in (a) and (b). (The difference will be substantial in an N-dimensional space with a large N. The limit of $N \to \infty$ is when we have continuous functions as the basis.)
(c) Given the non-orthogonal set of vectors, $\mathbf{d}_1 = (1, 1, 1)$, $\mathbf{d}_2 = (1, 1, 2)$, and $\mathbf{d}_3 = (2, 2, -5)$, consider the representation, $\mathbf{X} = b_1\, \mathbf{d}_1 + b_2\, \mathbf{d}_2 + b_3\, \mathbf{d}_3$, for the two cases: (i) $\mathbf{X} = (3, 5, 4)$, (ii) $\mathbf{X} = (1, 1, 0)$. In each case, determine if the expansion coefficients (b_1, b_2, b_3) exist and whether they are unique. If they exist but are not unique, find at least two examples of (b_1, b_2, b_3) that satisfy $\mathbf{X} = b_1\, \mathbf{d}_1 + b_2\, \mathbf{d}_2 + b_3\, \mathbf{d}_3$. Interpret your results geometrically.