MAE502 Spring 2013 Homework #6

Prob 1 (3 points)
For $u(x, t)$ defined on the infinite interval, $-\infty < x < \infty$, use the method of characteristics to find the solution of the PDE,

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0 \quad ,$$

with the boundary condition,

$$u(x, 0) = \mathrm{P}(x),$$

where

$$\begin{aligned} \mathrm{P}(x) &= 2 \quad , \quad \text{if } x \le 0 \\ &= 2 + x \quad , \quad \text{if } 0 < x \le 1 \\ &= 3 \quad , \quad \text{if } x > 1 \end{aligned}$$

Plot the solutions at $t = 0.1$ and 0.2, along with the initial state $u(x,0)$. Sketch the characteristics in the x-t plane. Also, what are the values of $u(x,t)$ at ($x = 3$, $t = 0.5$) and ($x = 100$, $t = 30$)?

Prob 2 (4 points)
For $u(x, t)$ defined on the infinite interval, $-\infty < x < \infty$, use the method of characteristics to find the solution of the PDE,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -t \quad ,$$

with the boundary condition,

$$u(x, 0) = \exp(-x^2) \ .$$

Plot the solution $u(x,t)$ at $t = 0.3$, along with the initial state $u(x,0)$. (It suffices to plot these two curves over the interval of $-3 \le x \le 3$.)

Prob 3 (2 points)
For $u(x, t)$ defined on the infinite interval, $-\infty < x < \infty$, use the method of characteristics to find the solution of the PDE,

$$\frac{\partial u}{\partial t} + t\, x \frac{\partial u}{\partial x} = 0 \quad ,$$

with the boundary condition,

$$u(x, 0) = \mathrm{P}(x).$$

Plot the solutions at $t = 1.0$ and 1.5, along with the initial state $u(x,0)$, for the following two cases:
(i) $\mathrm{P}(x) \equiv \exp(-x^2)$. (ii) $\mathrm{P}(x) \equiv \exp[-(x-1)^2]$.

Prob 4 (1.5 point)
For $u(t)$ defined on $t \geq 0$, solve the nonlinear ODE,

$$\frac{du}{dt} = u - u^2 \quad ,$$

with the initial condition,

$u(0) = A$. (A is a given constant.)

By setting the du/dt in the ODE to zero, we can readily deduce that there are two possible steady state solutions: $u(\infty) = 0$ and $u(\infty) = 1$. (i) Find an example of the value of A with which the solution $u(t)$ will approach the steady state solution, $u(\infty) = 1$, as $t \to \infty$. (ii) Find an example of the value of A with which the solution $u(t)$ will approach the steady state, $u(\infty) = 0$, as $t \to \infty$. (iii) Find an example of the value of A with which the solution $u(t)$ will approach neither of the steady states but will instead blow up at a finite time. In this case, predict the precise value of t at which the finite-time blow up occurs. Sketch your solutions for the three examples you provided in (i), (ii), and (iii).

Prob 5 (1.5 point)
For $u(x, t)$ defined on $0 \leq x \leq 1$ and $t \geq 0$, a PDE is given

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} + 5 \frac{\partial^3 u}{\partial x^3} - 2u \quad ,$$

with the boundary conditions,

(i) $u(0,t) = u(1,t)$
(ii) $u_x(0,t) = u_x(1,t)$
(iii) $u_{xx}(0,t) = u_{xx}(1,t)$
(iv) $u(x,0) = 3 + \cos(2\pi x)$.

If the "total energy" of the system is defined as

$$E(t) \equiv \int_0^1 u(x,t)\,dx \quad ,$$

evaluate $E(t)$ at $t = 0.5$.