MAE/MSE 502, Fall 2014 Homework #1

1 point ≈ 1% of your total score for this class

Please submit hard copy of your work. Electronic submission will not be accepted. Please provide the print out of computer codes used in the work. The rules for collaboration on homework will be released separately. Please always follow the rules.

Prob. 1 (3 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with the boundary conditions (be aware that the second condition is imposed on the derivative of u),

(i) $u(0, t) = 3$, (ii) $u_x(1, t) = 0$, and (iii) $u(x,0) = 2x^2 - 4x + 3$,

Also, find the steady state solution ($u(x,t)$ as $t \to \infty$) of the system. Plot the solution, $u(x,t)$, as a function of x at $t = 0$, 0.1, 0.3, 0.5, and 1.0, along with the steady state. Please collect all six curves in a single plot. (Major deduction without the plot. Hand-drawn figures are not acceptable.)

Note: If your solution is expressed as an infinite series, it is your job to determine the appropriate number of terms to keep in the series to ensure that the solution is accurate. As a useful measure, the solution at $t = 0$ should nearly match the given initial state in the 3rd boundary condition. If they do not match, either the solution is wrong or you have not retained enough terms in the series. This remark applies to all future homework problems that require the evaluation of an infinite series.

Prob. 2 (2 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with the boundary conditions,

(i) $u_x(0, t) = 0$, (ii) $u_x(1, t) = 0$, and (iii) $u(x,0) = 2x^3 - 3x^2 + 0.3\cos(4\pi x) + 2$.

Also, find the steady state solution ($u(x,t)$ as $t \to \infty$) of the system. Plot the solution, $u(x,t)$, as a function of x at $t = 0$, 0.005, 0.03, and 0.1, along with the steady state. Please collect all five curves in a single plot.

See additional note in page 3 for a tip on using Matlab to evaluate integrals. It might be useful for solving Prob. 1 and 2.

Prob. 3 (1.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, consider the PDE,

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} - 100\,u \quad ,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u(1, t) = 0$, and (iii) $u(x, 0) = \mathrm{P}(x)$.

Find the analytic solution of this system for the following two cases: (A) $\mathrm{P}(x) \equiv \sin(2\pi x) + \sin(4\pi x)$, (B) $\mathrm{P}(x) \equiv \sin(3\pi x)$. Discuss the behavior of the solution as $t \to \infty$ for both cases. (Major deduction if this discussion is missing.)

Prob. 4 (0.5 point)
For the heat transfer problem, the Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K\,\frac{\partial^2 u}{\partial \hat{x}^2} \quad , \; 0 \le \hat{x} \le L \;\; (L \text{ is the length of the "metal rod", in meters) and } \hat{t} \ge 0 \, , \quad (1)$$

where $\hat{t}$ and $\hat{x}$ are time in seconds and distance in meters, K is thermal diffusivity in m^2/s, and u is temperature. In our class, we usually consider the non-dimensionalized version of (1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , \; 0 \le x \le 1 \;\text{ and }\; t \ge 0 \;, \qquad (2)$$

where x is related to $\hat{x}$ by $\hat{x} = L\,x$. In (2), time is also non-dimensionalized by $\hat{t} = T\,t$, where T is a certain dimensional time scale, and so on.

(a) In order to claim that the non-dimensionalized system, (1), is equivalent to its dimensional counterpart, (2), the three parameters K, L, and T must satisfy a unique relation. First, find out what this relation is. (For the discussion in part (b) & (c), it is useful to write the relation as $T = f(K, L)$.)

(b) Suppose that the non-dimensionalized Heat equation, (2), is used to model the real world problem of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem?

(c) Same as (b), but suppose that (2) describes heat transfer along a wooden stick that is 0.3 meter long and made of pine wood ($K \approx 10^{-7}$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to? (We consider $t = 0.01$ because it is about the time when a significant redistribution of temperature begins to take place in the scenario described in Part (d).)

(d) Are the time scales you obtained in (b) and (c) consistent with daily experience? For instance, one can use a long wooden spoon to continuously stir a boiling pot of soup without getting one's hand burned. In contrast, the same practice would make one very uncomfortable if the spoon is made entirely of copper. Note that the time scale for cooking a pot of soup is about 10 minutes. The length of a big wooden spoon is about a foot, or 0.3 meter.

Additional note: How to use Matlab to numerically evaluate an integral

The simplest Matlab function for this purpose is perhaps **trapz**. It uses the trapezoidal method to evaluate an integral. For example, to numerically evaluate

$$I = \int_0^1 \sin(x)\, dx$$

with $\Delta x = 0.01$, we first construct the discretized arrays of the coordinate points and the values of the integrand at those points. We then call **trapz** with those two arrays as the input to complete the integration. The Matlab code is very simple:

```
x = [0:0.01:1];
y = sin(x);
Integ = trapz(x,y)
```

One can readily verify the outcome with the analytic result of $I = 1 - \cos(1)$.