MAE/MSE 502, Fall 2014 HW1

Prob 1 Solution

The steady state is $U(x) = 3$.

The full solution is

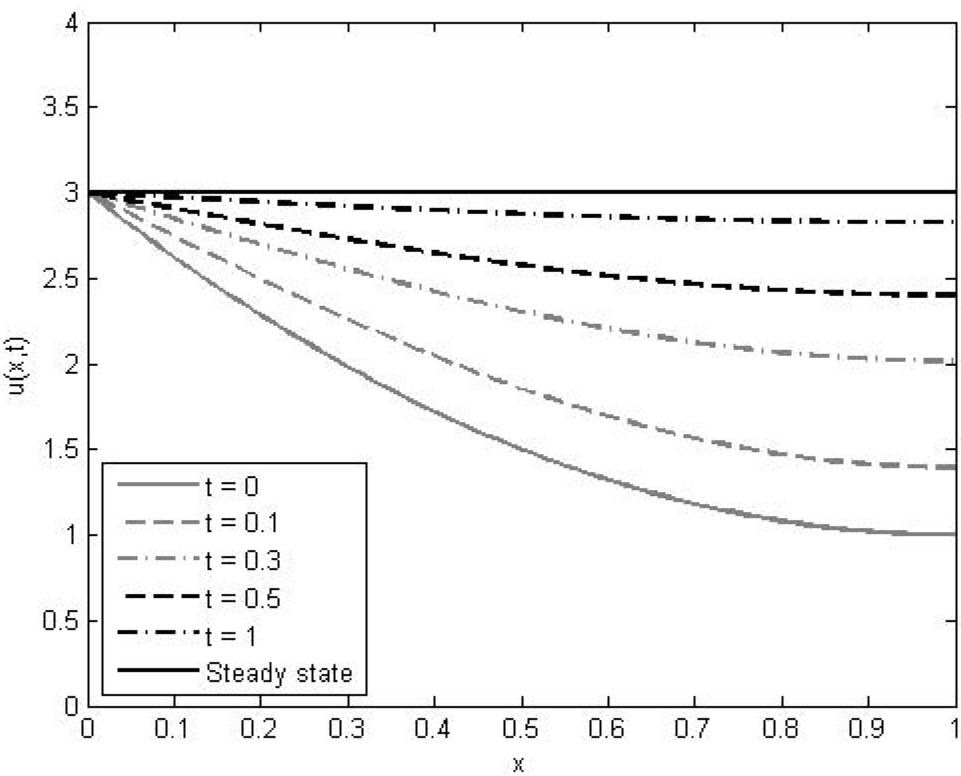
$$u(x,t) = 3 + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{2}\right) \exp\left(-\left(\frac{n\pi}{2}\right)^2 t\right) \quad ,$$

with the summation carried over <u>odd values of n only</u>, and

$$a_n = \frac{\int_0^1 (2x^2 - 4x)\sin\left(\frac{n\pi x}{2}\right) dx}{\int_0^1 \left[\sin\left(\frac{n\pi x}{2}\right)\right]^2 dx} \quad \text{, for } n = 1, 3, 5, 7, \ldots$$

The coefficient a_n can be evaluated numerically or analytically. For this problem, summation over a few leading terms would be accurate enough, since the initial condition is very smooth.

Plot of solution



Prob 1 Example of Matlab code

This example also shows how to make a gray-scale (black-and-white) line plot with distinctive line pattern and color.

```
clear
ntotal = 51; x = [0:0.001:1]; x1 = [0:0.01:1]; t5 = [0 0.1 0.3 0.5 1];
for n = 1:ntotal
    if mod(n,2) == 0
        a0(n) = 0;
    else
        integrand1 = sin(n*pi*x/2).*(2*x.^2-4*x);
        integrand2 = sin(n*pi*x/2).^2;
        a0(n) = trapz(x,integrand1)/trapz(x,integrand2);
    end
end
for it = 1:length(t5)
    t = t5(it);
    for ix = 1:length(x1)
        u(ix,it) = 0;
        for n = 1:ntotal
            if mod(n,2) == 1
                u(ix,it) = u(ix,it)+a0(n)*sin(n*pi*x1(ix)/2)*...
                    exp(-((n*pi/2)^2)*t);
            end
        end
        u(ix,it) = u(ix,it)+3;
    end
end
usteady = (x1-x1+1)*3;
hold on
plot(x1,u(:,1),'-','Color',[0.5 0.5 0.5],'LineWidth',2)
plot(x1,u(:,2),'--','Color',[0.5 0.5 0.5],'LineWidth',2)
plot(x1,u(:,3),'-.','Color',[0.5 0.5 0.5],'LineWidth',2)
plot(x1,u(:,4),'--','Color',[0 0 0],'LineWidth',2)
plot(x1,u(:,5),'-.','Color',[0 0 0],'LineWidth',2)
plot(x1,usteady,'-','Color',[0 0 0],'LineWidth',2)
xlabel('x');ylabel('u(x,t)')
legend('t = 0','t = 0.1','t = 0.3','t = 0.5','t = 1','Steady state',...
    'Location','SouthWest')
axis([0 1 0 4])
box on
hold off
```

Prob 2 Solution

The steady state is $U(x) = 1.5$. Note that for this problem one can obtain the steady state without solving the full $u(x,t)$. See the slides of Lecture #5 (especially the last two slides) for a relevant discussion.

The full solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\exp(-(n\pi)^2 t) \quad ,$$

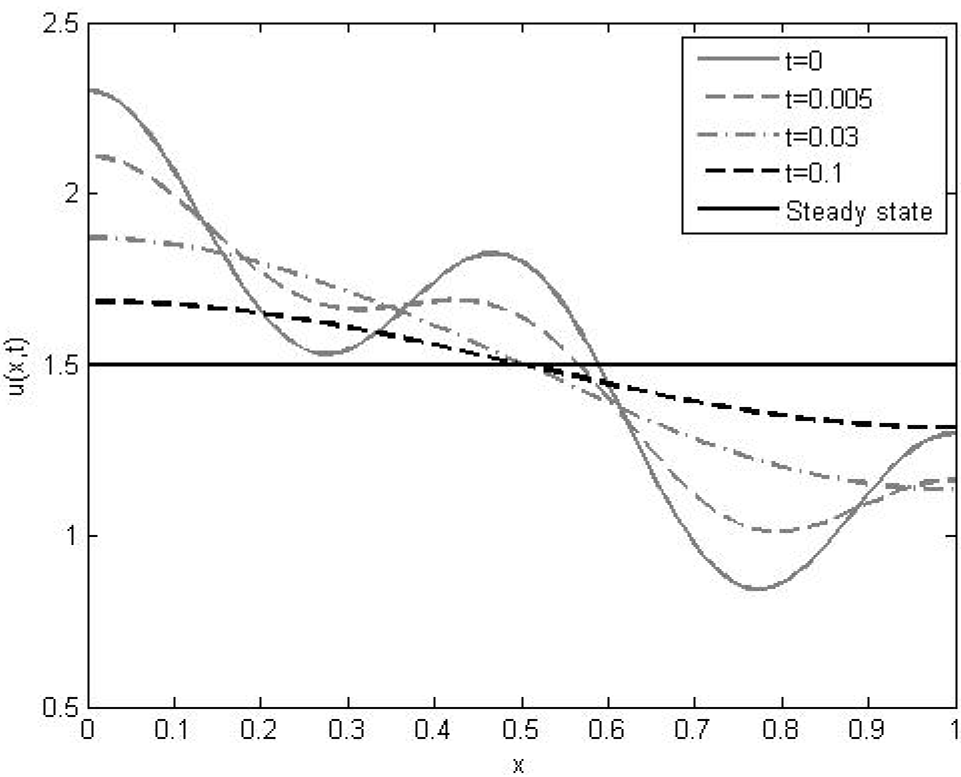
where

$$a_0 = \int_0^1 P(x)\, dx \ = 1.5 \,,$$

$$a_n = \frac{\int_0^1 P(x)\cos(n\pi x)\, dx}{\int_0^1 [\cos(n\pi x)]^2\, dx} \quad , \text{ for } n = 1, 2, 3, \ldots,$$

and $P(x) = 2x^3 - 3x^2 + 0.3\cos(4\pi x) + 2$.

Plot of solution



Prob 3 Solution

(A) $u(x, t) = \sin(2\pi x) \exp[(4\pi^2 - 100)\, t] + \sin(4\pi x) \exp[(16\pi^2 - 100)\, t]$

The solution blows up as $t \to \infty$ because the exponent in the second term, $(16\pi^2 - 100)$, is positive.

(B) $u(x, t) = \sin(3\pi x) \exp[(9\pi^2 - 100)\, t]$

The solution decays to zero (reaching a steady state of $U(x) = 0$) as $t \to \infty$ because the exponent, $(9\pi^2 - 100)$, is negative.

Prob 4 Solution

We will discuss the solution in more detail in class.