

## MAE/MSE502 Fall 2014 Homework #2

### Problem 1 (3 points)

For  $u(x, y)$  defined on the square domain of  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ , solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$

with the boundary conditions,

$$(i) u(0, y) = 0 \quad (ii) u(1, y) = 6(y - y^2) \quad (iii) u(x, 0) = 0 \quad (iv) u(x, 1) = 2 \sin(\pi x) + \sin(2\pi x) .$$

Make a contour plot of your solution. Based on the solution, evaluate  $u(x, y)$  at  $x = 0.4, y = 0.7$ . (Mandatory 0.5 point deduction if this value is not calculated.) See Additional Note at the end of this assignment for an example on how to make a contour plot using Matlab.

### Problem 2 (3 points)

For  $u(x, y)$  defined on the square domain of  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ , consider Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$

with the boundary conditions,

$$(i) u_y(x, 0) = 0 \quad (ii) u_x(0, y) = 0 \quad (iii) u_y(x, 1) = \cos(2\pi x) \quad (iv) u_x(1, y) = 0 .$$

(Note that all four conditions are imposed on the derivative of  $u$  in the direction normal to the boundary.)

(a) With the pure Neumann boundary conditions, Laplace's equation does not have a solution unless the solvability condition (cf. Eq. 2.5.61 in textbook)

$$\oint (\nabla u) \cdot \mathbf{n} \, dl = 0 \quad ,$$

is satisfied. (Here,  $\mathbf{n}$  is the outward unit normal vector at the boundary and the line integral is carried out along the entire boundary as a closed circuit.) Determine whether the solvability condition is satisfied for this system. Must provide a proof or a counterexample to support your "yes" or "no" answer. If your answer is "no", no need to proceed further.

(b) If your answer to Part (a) is "yes", it implies that the given system has multiple solutions. Provide two distinctive solutions for the system.

**Problem 3 (1.5 points)**

For  $u(x, t)$  defined on the domain of  $0 \leq x \leq 1$  and  $t \geq 0$ , solve the PDE,

$$\exp(-t) \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 100u \quad ,$$

with the boundary conditions,

$$(i) \ u(0, t) = 0 \quad (ii) \ u(1, t) = 0 \quad (iii) \ u(x, 0) = 4 \sin(3\pi x) + 7 \sin(4\pi x) \ .$$

Write out the analytic solution clearly. We expect a closed-form solution with no unevaluated integral or summation.

**Problem 4 (1.5 points)**

Consider the eigenvalue problem,

$$\frac{d^2 G}{dx^2} = c G \quad , \quad G(0) = 3 \quad , \quad G'(1) = 0 \quad . \quad (\text{Be aware that the 2nd b.c. is on the derivative of } u.)$$

**(a)** Determine the eigenvalues and the corresponding eigenfunctions of this problem.

Do consider all three possibilities with  $c > 0$ ,  $c = 0$ , and  $c < 0$ . Are the eigenvalues discrete? For example, if the boundary conditions are replaced by the familiar  $G(0) = 0$  and  $G(1) = 0$ , we would have  $c = c_n = -n^2 \pi^2$  ( $n$  is an integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation when the eigenvalues are not discrete is if all values within an interval,  $A \leq c \leq B$ , are valid eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

**(b)** Plot the eigenfunctions,  $G_c(x)$ , associated with the eigenvalues  $c = -50, -10, -1, 0, 1, 10$ , and  $50$ . (You will find in Part (a) that all those values are indeed valid eigenvalues.) Please collect all 7 curves in a single plot. Note that  $G(x)$  is defined only on the interval of  $0 \leq x \leq 1$ . Your plot should cover only that interval.

**(c)** Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^1 G_p(x) G_q(x) dx = 0, \quad \text{if } p \neq q \quad ,$$

where  $G_p(x)$  and  $G_q(x)$  are two eigenfunctions that correspond to two distinctive eigenvalues  $p$  and  $q$ ? Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of  $G_p(x)G_q(x)$  and show that it leads to a non-zero value even when  $p \neq q$ . One such counterexample would suffice to prove that the orthogonality relation does not hold. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all pairs of  $p$  and  $q$ .

**(d)** If  $G_p(x)$  is an eigenfunction corresponding to an eigenvalue,  $c = p$ , would  $AG_p(x)$  (where  $A$  is an arbitrary constant;  $A \neq 1$ ) also be an eigenfunction? Provide a brief explanation to support your yes/no answer.

## Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for  $u(x,y) = \sin(2\pi x)\exp(-2y)$  for the domain of  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ , using the contour levels of  $(-0.9, -0.7, -0.5, -0.3, -0.1, -0.05, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9)$ . The contours for  $u = -0.7, -0.3, 0.3,$  and  $0.7$  are labeled. It is essential to provide the coordinates of the grid ( $x2d$  and  $y2d$  in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of  $x$  and  $y$ . A black-and-white contour plot is acceptable as long as the contours are properly labeled.

```
clear
x = [0:0.01:1]; y = [0:0.01:1];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = sin(2*pi*x(i))*exp(-2*y(j));
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
[C,h] = contour(x2d,y2d,u, [-0.9:0.2:-0.1 -0.05 0.05 0.1:0.2:0.9]);
clabel(C,h, [-0.7 -0.3 0.3 0.7])
xlabel('x'); ylabel('y')
```

