MAE/MSE 502, Fall 2014, HW2 Solutions

Prob 1

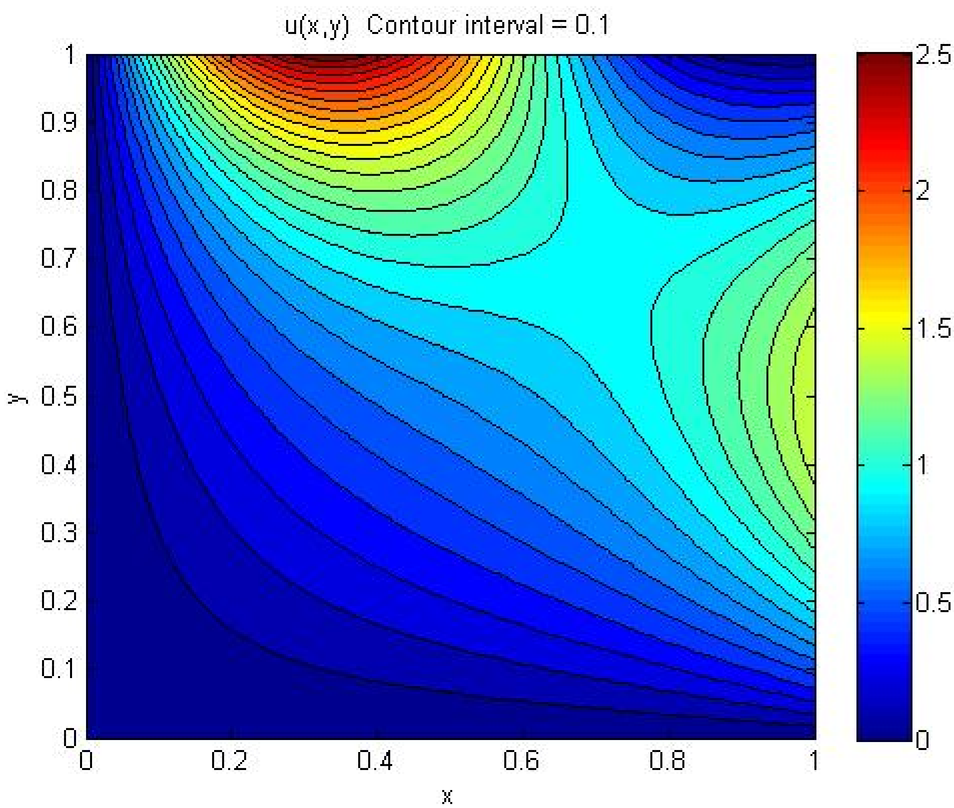
The solution is

$$u(x,y)=\frac{2}{\sinh(\pi)}\sin(\pi x)\sinh(\pi y)+\frac{1}{\sinh(2\pi)}\sin(2\pi x)\sinh(2\pi y)+\sum_{n=1}^{\infty}a_n\sinh(n\pi x)\sin(n\pi y)$$

where

$$a_n=\frac{1}{\sinh(n\pi)}\frac{\int_0^1 6(y-y^2)\sin(n\pi y)\,dy}{\int_0^1[\sin(n\pi y)]^2\,dy}\quad .$$

Plot:



$u(0.4, 0.7) = 0.9978$
(This value might vary slightly depending on the number of terms in the infinite series that you keep.)

Prob 1 Example of Matlab code

```
clear
x = [0:0.01:1];
y = [0:0.01:1];
ntrunc = 30;
for n = 1:ntrunc
    g = 6*(y-y.^2).*sin(n*pi*y);
    h = sin(n*pi*y).^2;
    a(n) = (1/sinh(n*pi))*trapz(y,g)/trapz(y,h);
end
for i = 1:101
for j = 1:101
   x2d(i,j) = x(i);
   y2d(i,j) = y(j);
   u1(i,j) = 0;
   for n = 1:ntrunc
       u1(i,j) = u1(i,j)+a(n)*sinh(n*pi*x(i))*sin(n*pi*y(j));
   end
   u2(i,j) =   (2/sinh(pi))*sin(pi*x(i))*sinh(pi*y(j)) + ...
             (1/sinh(2*pi))*sin(2*pi*x(i))*sinh(2*pi*y(j));
   u(i,j) = u1(i,j)+u2(i,j);
end
end
u(41,71)
contourf(x2d,y2d,u,[0:0.1:2.5]);
colorbar
xlabel('x');ylabel('y');title('u(x,y)  Contour interval = 0.1')
```

Prob 2

(a) The solvability condition is satisfied since

$$\oint (\nabla u)\cdot \boldsymbol{n}\, dl = \int_0^1 \cos(2\pi x)\, dx = 0 \quad .$$

(b) The solutions are

$$u(x, y) = B + \frac{\cos(2\pi x)\cosh(2\pi y)}{2\pi \sinh(2\pi)} \quad ,$$

where B can be any constant.   Choosing two different values of B (e.g., $B = 1$ and $B = 1{,}000{,}000$) would produce two distinctive solutions.

Prob 3

The solution is

$$u(x, t) = 4 \exp[-(9\pi^2 + 100)(\exp(t) - 1)] \sin(3\pi x) + 7 \exp[-(16\pi^2 + 100)(\exp(t) - 1)] \sin(4\pi x)$$

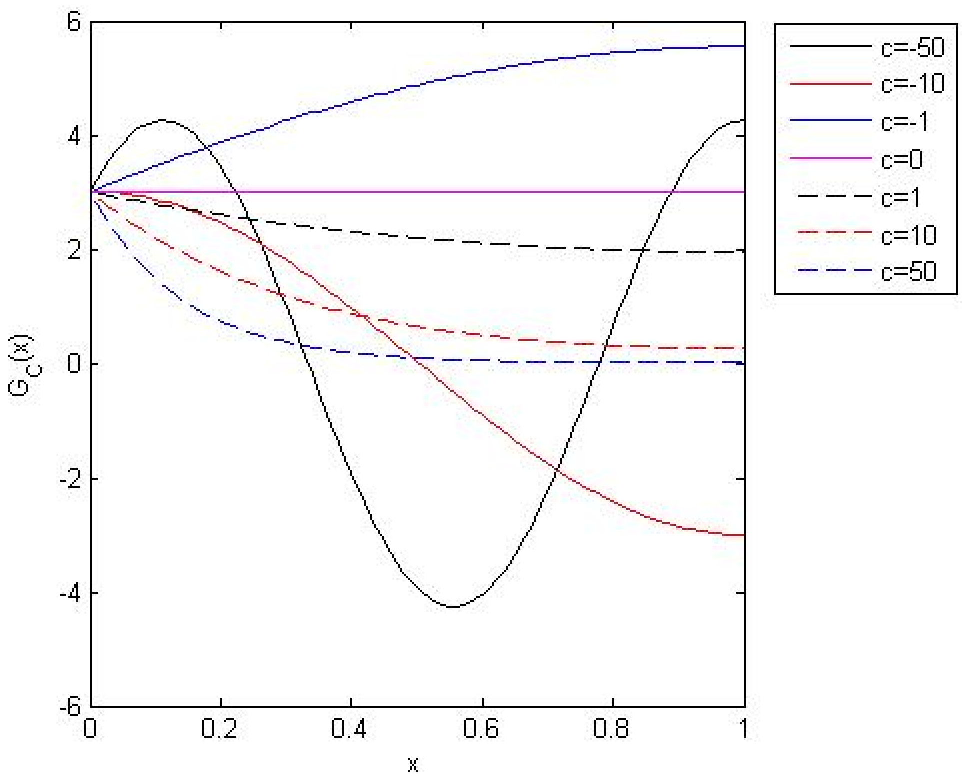
Prob 4

(a) All values of c on the real axis ($-\infty < c < \infty$), except $c = -(n\pi/2)^2$ with $n = 1, 3, 5, \ldots$, are eigenvalues. (At those special discrete values of c, the solution blows up.) The eigenfunctions are

$$G_c(x) = \frac{3\cos(\sqrt{-c}\,(x-1))}{\cos(\sqrt{-c})} \quad \text{, if } c < 0$$
$$G_c(x) = 3 \quad \text{, if } c = 0$$
$$G_c(x) = \frac{3\cosh(\sqrt{c}\,(x-1))}{\cosh(\sqrt{c})} \quad \text{, if } c > 0 \,.$$

Note that we have written the eigenfunctions in a slightly more compact form. For example, for $c > 0$, the eigenfunction can also be written as $G_c(x) = 3[\cosh(\sqrt{c}\,x) - \sinh(\sqrt{c}\,x)\tanh(\sqrt{c})]$, and so on.

(b) Plot:



(c) The orthogonality relation is not satisfied by the eigenfunctions. For example, all eigenfunctions corresponding to $c \geq 0$ are positive definite. Picking any two eigenfunctions, $G_p(x)$ and $G_q(x)$ (corresponding to $c = p$ and $c = q$), from that group, the integral of their product from 0 to 1 is guaranteed to be positive and not zero.

(d) No. If $G_p(x)$ is an eigenfunction, it satisfies the first b.c. so $G_p(0) = 3$. This means $AG_p(0) = 3A \neq 3$ (unless $A = 1$, which is the trivial case that we ruled out in the question). In other words, $AG_p(x)$ violates the first b.c. and cannot be an eigenfunction.