MAE/MSE 502, Fall 2014, Homework #3

Prob 1 (3 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the 1-D Wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ ,$$

with the boundary conditions,

(i) $u(0, t) = 0$
(ii) $u(1, t) = 0$
(iii) $u(x, 0) = \mathrm{P}(x)$
(iv) $u_t(x, 0) = 0$ (u_t is $\partial u/\partial t$) ,

where

$$\begin{aligned} \mathrm{P}(x) &= -x &&, \text{ if } 0 \le x \le 0.8 \\ &= 4(x-1) &&, \text{ if } 0.8 < x \le 1 \ . \end{aligned}$$

Plot the solution at $t = 0, 0.3, 0.5, 0.7, 1.0$, and 1.8. Please collect all 6 curves in one figure.

Prob 2 (3 points)

For $u(x,t)$ defined on the domain of $2 \le x \le 5$ and $t \ge 0$, consider the following PDE and boundary conditions:

$$e^{-2x} \frac{\partial u}{\partial t} = 0.5 \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + x^2 e^{-2x} u \ ,$$

(i) $u(2, t) = 3\, u_x(2, t)$, (u_x is $\partial u/\partial x$)
(ii) $u(5, t) = 0$,
(iii) $u(x,0) = \mathrm{P}(x)$,

where $\mathrm{P}(x)$ is a well-behaved function. (The detail of $\mathrm{P}(x)$ is unimportant for this problem.)
(a) Perform separation of variables on the PDE and the first two boundary conditions to obtain an eigenvalue problem in the x-direction (including an ODE and two boundary conditions) and an accompanying ODE in the t-direction. Write them out clearly. Is the eigenvalue problem in the x-direction of the form of a Sturm-Liouville system?
(b) If your answer to the last question in Part (a) is "no", no need to proceed further. If the answer is "yes", and if c_n and $G_n(x)$ denote the n-th eigenvalue and n-th eigenfunction of the eigenvalue problem in the x-direction, express the final solution, $u(x,t)$, as an eigenfunction expansion in terms of c_n and $G_n(x)$. Please also provide the formula for evaluating the expansion coefficients when $\mathrm{P}(x)$ is given.