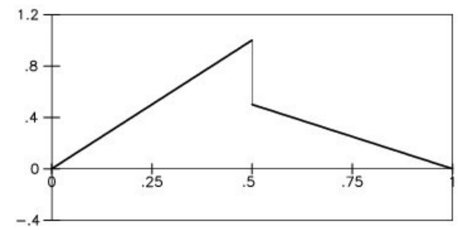
MAE/MSE 502 Fall 2014, Homework #4

Prob. 1 (4 points)

(a) For a function defined on the interval of $0 \le x \le 1$ given as



$$f(x) = 2x, \quad 0 \le x \le 1/2$$
$$\quad\; = 1 - x, \quad 1/2 < x \le 1,$$

work out its Fourier Cosine series expansion,

$$F(x) \approx F_C(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x) \quad ,$$

where $F(x)$ is the even extension of $f(x)$ and $F_C(x)$ denotes the Fourier Cosine series representation of $F(x)$. A sketch of $f(x)$ is shown at right; Notice a discontinuity at $x = 1/2$.

(b) Plot the original $f(x)$ and its Fourier Cosine series representation, $F_C(x)$, truncated (inclusively) at $n = 5$, 10, and 30. Please collect all four curves in a single plot. What are the values of $F_C(x)$ at $x = 0.35$ for the three cases truncated at $n = 5$, 10, and 30? Compare them to the exact value, $f(0.35)$, to determine the percentage error (using the exact value as denominator) for the three cases. Repeat the exercise for $x = 0.49$ (a point close to the discontinuity). Discuss the results.

(c) Define $S(\mathrm{N})$ as the value of $F_C(1/2)$ calculated from the Fourier Cosine series truncated (inclusively) at $n = \mathrm{N}$, plot $S(\mathrm{N})$ as a function of N over the range of $1 \le \mathrm{N} \le 30$. What value does $S(\mathrm{N})$ converge to at large N?

Prob. 2 (3 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^5 u}{\partial x^5} + \frac{\partial^2 u}{\partial x^2} \quad ,$$

with the boundary conditions (the first five simply indicate that the system is periodic in x),

(i) $u(0, t) = u(2\pi, t)$
(ii) $u_x(0, t) = u_x(2\pi, t)$
(iii) $u_{xx}(0, t) = u_{xx}(2\pi, t)$
(iv) $u_{xxx}(0, t) = u_{xxx}(2\pi, t)$
(v) $u_{xxxx}(0, t) = u_{xxxx}(2\pi, t)$
(v) $u(x, 0) = 1 + \sin(2x) + \cos(3x)$.

We expect a closed-form solution without any unevaluated integral or summation of infinite series.

Prob. 3 (3 points)
(a) For $u(x,y,t)$ defined on the domain of $0 \le x \le 1$, $0 \le y \le 1$, and $t \ge 0$, solve the modified two-dimensional heat equation (be aware of a factor of 4 in the last term),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4\,\frac{\partial^2 u}{\partial y^2} \quad ,$$

with the boundary conditions

(i) $u(0, y, t) = 0$
(ii) $u(1, y, t) = 0$
(iii) $u(x, 0, t) = 0$
(iv) $u(x, 1, t) = 0$
(v) $u(x, y, 0) = \sin(2\pi\, x)\,\sin(3\pi\, y)$.

We expect a closed-form solution without any unevaluated integral or summation of infinite series.