MAE502 Spring 2014 Homework #1

1 point ≈ 1% of your total score for this class

Collaboration is allowed but must be properly acknowledged. Please submit hard copy of your work on the due date <u>before class</u>. Electronic submission will not be accepted. Please provide the print out of Matlab (or Fortran, C++, etc.) codes used in the work.

Problem 1 (3 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u(1, t) = 0$, and (iii) $u(x,0) = P(x)$,

where

$$P(x) \equiv 8x - 16x^2 , \text{ if } 0 \le x \le 0.5$$
$$\equiv 0 \quad , \text{ if } 0.5 < x \le 1 .$$

(See Fig. 1 for a plot of P(x).)

Plot the solution as a function of x at $t = 0$, 0.01, 0.05, and 0.2. Please collect all four curves in a single plot. (Major deduction without the plot. <u>Hand-drawn figures are not acceptable</u>.) What is the value of $u(x,t)$ at $x = 0.6$, $t = 0.05$? What is the steady state solution ($u(x,t)$ as $t \to \infty$) for this system?

Note: If your solution is expressed as an infinite series, it is your job to determine the appropriate number of terms to keep in the series to ensure that the solution is accurate. As a useful measure, the solution at $t = 0$ should nearly match the given initial state, P(x), in the 3rd boundary condition. If they do not match, either the solution is wrong or you have not retained enough terms in the series.

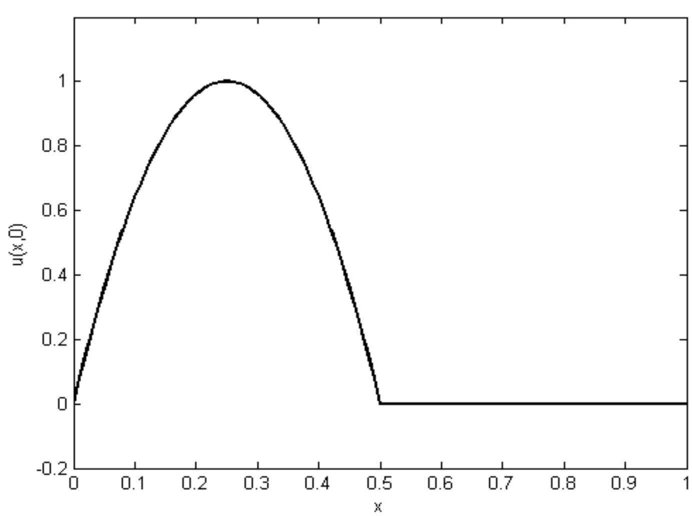

Fig. 1

Problem 2 (2 points)
Consider the system described in Prob 1 but with the original boundary conditions replaced by the periodic boundary conditions:

(i) $u(0,t) = u(1,t)$,
(ii) $u_x(0,t) = u_x(1,t)$ (u_x is $\partial u/\partial x$).

Find the solution of this new system and plot it as a function of x at $t = 0$, 0.01, 0.05, and 0.2. Please collect all 4 curves in one plot. Compare the solution with its counterpart in Prob 1 and comment on the effect of the boundary conditions on the solution. What is the steady state solution ($u(x,t)$ as $t \to \infty$) for this system?

Problem 3 (1.5 points)
Consider the system described in Prob 1 but with the PDE changed to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 10u \quad .$$

Find the solution of this new system and plot it as a function of x at $t = 0$, 0.01, and 0.05. Please collect all 3 curves in one plot. Compare the solution with its counterpart in Prob 1 and comment on the effect of the added term ($-10\,u$) on the solution.

Problem 4 (1 point)
For $G(x)$ defined on the interval of $2 \le x \le 5$, solve the eigenvalue problem:

$$\frac{d^2 G}{d x^2} = c\,G \quad ,$$

with the boundary conditions,

(i) $G(2) = 0$,
(ii) $G'(5) = G'(2)$ ($G'(x)$ is dG/dx) .

Please state clearly what the eigenvalues and corresponding eigenfunctions are. Plot the three eigenfunctions corresponding to the three eigenvalues with the smallest absolute values. In the plot, adjust the "amplitude" of the eigenfunctions such that the maximum of $|G(x)|$ is 1 for each individual eigenfunction.

Problem 5 (0.5 point)

For the heat transfer problem, the Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K \frac{\partial^2 u}{\partial \hat{x}^2} \quad , \; 0 \le \hat{x} \le L \quad (L \text{ is the length of the "metal rod", in meters}) \text{ and } \hat{t} \ge 0 \, , \quad (1)$$

where $\hat{t}$ and $\hat{x}$ are time in seconds and distance in meters, K is thermal diffusivity in m^2/s, and u is temperature. In our class, we usually consider the non-dimensionalized version of (1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , \; 0 \le x \le 1 \text{ and } t \ge 0 \; , \qquad (2)$$

where x is related to $\hat{x}$ by $\hat{x} = L\,x$. In (2), time is also non-dimensionalized by $\hat{t} = T\,t$, where T is a certain dimensional time scale, and so on.

(a) In order to claim that the non-dimensionalized system, (1), is equivalent to its dimensional counterpart, (2), the three parameters K, L, and T must satisfy a unique relation. First, find out what this relation is. (For the discussion in part (b) & (c), it is useful to write the relation as $T = f(K, L)$.)

(b) Suppose that the non-dimensionalized Heat equation, (2), is used to model the real world problem of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem?

(c) Same as (b), but suppose that (2) describes heat transfer along a wooden stick that is 0.3 meter long and made of pine wood ($K \approx 10^{-7}$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to? (We consider $t = 0.01$ because it is about the time when a significant redistribution of temperature begins to take place in the scenario described in Part (d).)

(d) Are the time scales you obtained in (b) and (c) consistent with daily experience? For instance, one can use a long wooden spoon to continuously stir a boiling pot of soup without getting one's hand burned. In contrast, the same practice would make one very uncomfortable if the spoon is made entirely of copper. Note that the time scale for cooking a pot of soup is about 10 minutes. The length of a big wooden spoon is about a foot, or 0.3 meter.