

MAE/MSE502 Spring 2014 Homework #2

Problem 1 (3 points)

For $u(x,t)$ defined on the domain of $0 \leq x \leq 1$ and $t \geq 0$, solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with boundary conditions,

- (i) $u(0, t) = 2$
- (ii) $u_x(1, t) = 2$ (u_x is $\partial u / \partial x$)
- (iii) $u(x, 0) = 3x^2 - 4x + 2$

Plot the solution as a function of x , at $t = 0, 0.05, 0.2, \text{ and } 0.5$, along with the steady state solution. Please collect all 5 curves in one plot. (Note: As always, it is your job to determine the appropriate number of terms to keep in the infinite series to ensure that the solution is accurate. As a useful measure, the solution at $t = 0$ should match the given initial state for $u(x, 0)$ in the 3rd boundary condition.)

Problem 2 (4 points)

For $u(x, y)$ defined on the rectangular domain of $0 \leq x \leq 1$ and $0 \leq y \leq 2$, solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ,$$

with the boundary conditions,

- (i) $u(0, y) = 0$
- (ii) $u(1, y) = 3y - 1.5y^2$
- (iii) $u(x, 0) = [\sin(\pi x)]^2$
- (iv) $u(x, 2) = 0$

Make a contour plot of your solution. The recommended contour levels are (0.1, 0.2, 0.3, ..., 1.5). What is the value of $u(x, y)$ at $x = 0.5, y = 1.5$? (Mandatory 1 point deduction if this value is not calculated.) See additional note at the end of this assignment for an example on how to make a contour plot using Matlab.

Problem 3 (1 point)

Consider the following eigenvalue problem,

$$\frac{d^2 G}{dx^2} = c G, \quad G(0) = 0, \quad G(1) = 2$$

(a) Determine the eigenvalues and the corresponding eigenfunctions of this problem. Do consider all three possibilities with $c > 0$, $c = 0$, and $c < 0$. Are the eigenvalues discrete? For example, if the boundary conditions are replaced by the familiar $G(0) = 0$ and $G(1) = 0$, we would have $c = c_n = -n^2 \pi^2$ (n is an integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation when the eigenvalues are not discrete is if all values within an interval, $A \leq c \leq B$, are valid eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

(b) Plot the eigenfunctions, $G_c(x)$, associated with the eigenvalues $c = -50, -20, -5, 0, 5, 20$, and 50 . (You will find in Part (a) that all those values are indeed valid eigenvalues.) Please collect all 7 curves in a single plot. Note that $G(x)$ is defined only on the interval of $0 \leq x \leq 1$. Your plot should cover only that interval.

(c) Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^1 G_p(x) G_q(x) dx = 0, \quad \text{if } p \neq q,$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions that correspond to two distinctive eigenvalues p and q ? Please provide a concrete discussion to support your conclusion. Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of $G_p(x)G_q(x)$ and show that it leads to a non-zero value even when $p \neq q$. One such counterexample would suffice to prove that the orthogonality relation does not hold. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all pairs of p and q .

(d) If $G_p(x)$ is an eigenfunction corresponding to an eigenvalue, $c = p$, would $AG_p(x)$ (where A is an arbitrary constant; $A \neq 1$) also be an eigenfunction? Provide a brief explanation to support your yes/no answer.

Additional note for Prob 2: Matlab example

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x)\exp(-2y)$ for the domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, using the contour levels of $(-0.9, -0.7, -0.5, -0.3, -0.1, -0.05, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9)$. The contours for $u = -0.7, -0.3, 0.3,$ and 0.7 are labeled. It is essential to provide the coordinates of the grid ($x2d$ and $y2d$ in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y . A black-and-white contour plot is acceptable as long as the contours are properly labeled.

```
clear
x = [0:0.01:1]; y = [0:0.01:1];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = sin(2*pi*x(i))*exp(-2*y(j));
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
[C,h] = contour(x2d,y2d,u,[-0.9:0.2:-0.1 -0.05 0.05 0.1:0.2:0.9]);
clabel(C,h,[-0.7 -0.3 0.3 0.7])
xlabel('x'); ylabel('y')
```

