MAE/MSE502 Spring 2014 Homework #3

Problem 1 (3.5 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 5$ and $t \ge 0$, solve the Wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad ,$$

with the boundary conditions,

(i) $u(0, t) = 0$
(ii) $u(5, t) = 0$
(iii) $u(x, 0) = \mathrm{P}(x)$
(iv) $u_t(x, 0) = 0$ $\quad$ (u_t is $\partial u/\partial t$) ,

where

$$\mathrm{P}(x) = x \quad , \text{ if } 0 \le x \le 1$$
$$= (5 - x)/4 \quad , \text{ if } 1 < x \le 5 \; .$$

Plot the solution as a function of x at $t = 0$, 1.5, 2.5, 3.5, 5, and 9. Please collect all 6 curves in one plot.

Problem 2 (2.5 points)

For $u(x, y)$ defined on the domain of $0 \le x \le 1$ and $0 \le y \le 1$, solve Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$

with the boundary conditions,

(i) $u(x, 0) = 2 + \cos(\pi x)$
(ii) $u_x(0, y) = 0$
(iii) $u_y(x, 1) = 0$
(iv) $u_x(1, y) = 0$

Write out the <u>analytic solution</u> clearly and evaluate $u(x,y)$ at $x = 0.3$, $y = 0.4$. Make a contour plot of the solution.

Note: The boundary condition for Prob 2 is not of purely Dirichlet or purely Neumann type, but is "mixed". In this case, a solution may exist depending on the detail of the boundary condition. If the system in Prob 2 describes a 2-D heat transfer problem, the boundary conditions (ii)-(iv) imply that the top, left, and right boundaries are all thermally insulated (i.e., no heat flux can go through those boundaries).

Problem 3 (1 point)

A system of Laplace's equation with a purely Neumann boundary condition does not have a solution unless the line integral of $(\nabla u)\cdot\boldsymbol{n}$ (where $\boldsymbol{n}$ is the outward unit normal vector at the boundary) along the entire boundary vanishes,

$$\oint (\nabla u)\cdot\boldsymbol{n}\, dl = 0 \qquad \text{Eq. (1)} .$$

Equation (1) is called the solvability condition (cf. Eq. 2.5.61 in textbook and informal slide sets for Lecture #8 and #9). Consider the following system of Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ,$$

with the boundary conditions,

(i) $u_y(x, 0) = 2$
(ii) $u_x(0, y) = 0$
(iii) $u_y(x, 1) = 2$
(iv) $u_x(1, y) = 0$.

(a) Is the solvability condition satisfied for this system? If your answer is "no", prove that the line integral in Eq. (1) is nonzero. You do not have to proceed further. If your answer is "yes", proceed to Part (b).

(b) Find the solution(s) of the system. Is the solution unique? If your answer is "yes", prove the uniqueness of the solution. If your answer is "no", provide an example of two distinctive solutions for the system.