MAE502 Spring 2014 Homework #4B

Problem 1 (5 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, consider the Heat equation with a non-constant thermal diffusivity and non-constant dissipation,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}[K(x)\frac{\partial u}{\partial x}] - D(x)u \quad ,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u(1, t) = 0$, and (iii) $u(x,0) = \mathrm{P}(x)$,

where

$$\mathrm{P}(x) \equiv 8x - 16x^2 \ , \ \text{if } 0 \le x \le 0.5$$
$$\equiv 0 \ , \ \text{if } 0.5 < x \le 1 \ .$$

Note that if one sets $K(x) \equiv 1$ and $D(x) \equiv 0$, the system described above will be identical to the one in HW1-Prob 1. With a non-constant $K(x)$, the approach of separation of variables and eigenfunction expansion will still work because the ODE and b.c.'s in the x-direction obtained by separation of variables will form a Sturm-Liouville system. You are required to solve the problem using the following approach (*cf.* Sec 5.4): (i) Solve the Sturm-Liouville system to obtain the eigenvalues and eigenfunctions, (ii) Expand $u(x,t)$ in terms of the eigenfunctions in x (and their counterparts in the t-direction), (iii) Find the expansion coefficients by applying the orthogonality relation of the eigenfunctions. With a non-constant $K(x)$, the eigenfunctions will no longer be sine and cosine. Although one might still be able to solve them analytically (which is allowed), it is recommended that the eigenvalues and eigenfunctions be obtained numerically by the finite difference method. Additional instructions on the technical detail will be given in class.

(a) Using the approach of separation of variables and eigenfunction expansion, solve the case with $K(x) = 5x^2 + 0.01$, $D(x) = 0$. Plot the solution as a function of x at $t = 0$, 0.01, 0.05, and 0.2. To receive full credit, you must also provide:

(I) The five eigenvalues with the <u>smallest absolute values</u>.
(II) A plot of the five eigenfunctions corresponding to the five eigenvalues given in (I). Please collect all five curves in one figure.

The eigenvalues and eigenfunctions in (I) and (II) refer to those obtained by solving the Sturm-Liouville system in the x-direction. The eigenfunctions will be functions of x only.

(b) Repeat (a) but solve the case with $K(x) = 5x^2 + 0.01$, $D(x) = 50x$. Plot the solution as a function of x at $t = 0$, 0.01, 0.05, and 0.2, and provide the additional information described in (I) and (II) in Part (a).

Note: Although you are asked to show only five eigenfunctions, to obtain an accurate solution more than five should be included in the expansion. The recommended minimum resolution for this problem is $\Delta x = 0.01$ in the finite difference scheme for the eigenvalue problem, and 40 eigenfunctions in the eigenfunction expansion. As a check of the accuracy of your solution, the solution at $t = 0$ should closely match the $\mathrm{P}(x)$ given in the 3rd b.c.