MAE/MSE502 Spring 2014 Homework #5

Prob. 1 (2.5 points)
Consider the function (see sketch below) defined on the interval of $0 \le x \le 1$,

$$f(x) = 1, \quad 0 \le x \le 0.5$$
$$= 1 - x, \quad 0.5 < x \le 1 \ .$$

(a) Work out the Fourier Sine series expansion,

$$F_S(x) \approx \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad ,$$

where $F_S(x)$ denotes the Fourier Sine series representation of $f(x)$. Plot the original $f(x)$ and its Fourier Sine series representation, $F_S(x)$, truncated (inclusively) at n = 5, 10, and 30. Please collect all four curves in a single plot.

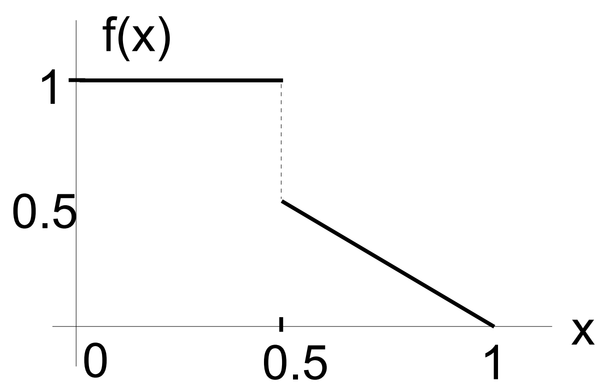
(b) What are the values of $F_S(x)$ at x = 0.75 for the three cases truncated at n = 5, 10, and 30? Compare them to the exact value, $f(0.75)$, to determine the percentage error (using the exact value as denominator) for the three cases. Repeat the exercise for x = 0.51 (a point close to the discontinuity). Discuss the results.

(c) Define S(N) as the value of $F_S(0.5)$ calculated from the Fourier Sine series truncated at n = N, plot S(N) as a function of N for the range of $1 \le \text{N} \le 30$. What value does S(N) converge to at large N?

(d) Repeat (a) but now work out the Fourier Cosine series expansion,

$$F_C(x) \approx \sum_{n=0}^{\infty} a_n \cos(n\pi x) \quad ,$$

where $F_C(x)$ denotes the Fourier Cosine series representation of $f(x)$. (Beware that the summation starts at n = 0.) Plot the $F_C(x)$ truncated (inclusively) at n = 5, 10, and 30, along with the original $f(x)$.



Prob. 2 (3.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \frac{\partial^4 u}{\partial x^4} \quad ,$$

with the boundary conditions (the first four simply indicate that the system is periodic in x),

(i) $u(0, t) = u(2\pi, t)$
(ii) $u_x(0, t) = u_x(2\pi, t)$
(iii) $u_{xx}(0, t) = u_{xx}(2\pi, t)$
(iv) $u_{xxx}(0, t) = u_{xxx}(2\pi, t)$
(v) $u(x, 0) = 5 + 2\cos(3x)$.

Evaluate $u(x, t)$ at $x = 1$, $t = 0.01$.

Prob. 3 (1 point)
(a) Given the following function defined on the semi-infinite interval, $0 \le x < \infty$,

$$f(x) = 1 \;, \text{ if } 0 \le x \le 1, \qquad \text{Eq. (1)}$$
$$= 0 \;, \text{ if } x > 1 \;,$$

determine the Fourier Sine transform of $f(x)$, $F(\omega)$, that satisfies

$$f(x) = \int_0^\infty F(\omega) \sin(\omega x)\, d\omega \quad .$$

Plot $F(\omega)$ as a function of ω for the range $0 \le \omega \le 30$.

(b) If the $f(x)$ in Eq. (1) is instead defined on a finite interval, $0 \le x \le L$ (but otherwise retains its definition in Eq. (1), i.e., $f(x) = 0$ if $1 < x \le L$), find the coefficients, a_n, for the Fourier Sine series of $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n \pi x}{L}\right) \quad .$$

Plot a_n as a function of n for the following cases: (i) For $L = 2$, plot a_n over the range of $1 \le n < 60/\pi$. (ii) For $L = 5$, plot a_n for $1 \le n < 150/\pi$. (iii) For $L = 100$, plot a_n for $1 \le n < 3000/\pi$. Compare these plots with the plot of $F(\omega)$ in (a). <u>Discuss your results</u>.

(Note: This homework illustrates the correspondence between Fourier series and Fourier integral. When making the plots, beware that the "n" in Part (b) is an integer while the "ω" in Part (a) can be any real number.)