

## MAE/MSE502 Fall 2015 Homework #2

### Prob. 1 (3 points)

(a) For  $u(x, t)$  defined on the domain of  $0 \leq x \leq 1$  and  $t \geq 0$ , solve the PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \cos(\pi x)e^{-t} + \frac{1}{1+t} ,$$

with the boundary conditions

- (i)  $u_x(0, t) = 0$
- (ii)  $u_x(1, t) = 0$
- (iii)  $u(x, 0) = 2 + \cos(\pi x) + \cos(2\pi x)$ .

We expect a closed-form solution without any unevaluated integral or summation of infinite series.

(b) Does the system in (a) have a steady state solution? If yes, find the steady state. If no, explain why. Does the system in (a) have a steady state if the last term in the right hand side of the PDE is replaced by  $1/(1+t)^2$ ? If yes, find the steady state. If no, explain why.

### Prob 2 (3 points)

Consider the eigenvalue problem,

$$\frac{d^2 G}{dx^2} + 2 \frac{dG}{dx} = c G , \quad G(0) = 0 , \quad G(1) = 0 .$$

(a) Determine the eigenvalues and the corresponding eigenfunctions of this problem.

(b) Plot the three eigenfunctions associated with the three eigenvalues with the smallest absolute values. Since the amplitude of an eigenfunction can be arbitrary for this problem, it is recommended that in the plot each eigenfunction be normalized such that

$$\int_0^1 [G(x)]^2 dx = 1 .$$

(c) Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^1 G_p(x)G_q(x) dx = 0 , \quad \text{if } p \neq q ,$$

where  $G_p(x)$  and  $G_q(x)$  are two eigenfunctions associated with two distinctive eigenvalues  $p$  and  $q$ ? Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of  $G_p(x)G_q(x)$  and show that it leads to a non-zero value even when  $p \neq q$ . One such counterexample would suffice to prove that the orthogonality relation does not hold. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all pairs of  $p$  and  $q$ .

**Prob 3** (4 points)

For  $u(x, y)$  defined on the square domain of  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ , solve the slightly modified Laplace's equation

$$2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ,$$

with the boundary conditions,

$$(i) u(0, y) = 0 \quad (ii) u(1, y) = y - y^2 \quad (iii) u(x, 0) = 0 \quad (iv) u(x, 1) = \sqrt{x} - x .$$

Make a contour plot of your solution. Based on the solution, evaluate  $u(x, y)$  at  $x = 0.4, y = 0.6$ . (Major deduction if this value is not calculated.) See Additional Note below for an example of Matlab code for making a contour plot.

**Additional Note: Using Matlab to make a contour plot**

The following Matlab code makes a contour plot for  $u(x,y) = \sin(2\pi x)\exp(-2y)$  for the domain of  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ , using the contour levels of  $(-0.9, -0.7, -0.5, -0.3, -0.1, -0.05, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9)$ . The contours for  $u = -0.7, -0.3, 0.3$ , and  $0.7$  are labeled. It is essential to provide the coordinates of the grid ( $x2d$  and  $y2d$  in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of  $x$  and  $y$ . A black-and-white contour plot is acceptable as long as the contours are properly labeled. For Prob 3, the recommended contour interval is 0.03.

```
clear
x = [0:0.01:1]; y = [0:0.01:1];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = sin(2*pi*x(i))*exp(-2*y(j));
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
[C,h] = contour(x2d,y2d,u, [-0.9:0.2:-0.1 -0.05 0.05 0.1:0.2:0.9]);
clabel(C,h, [-0.7 -0.3 0.3 0.7])
xlabel('x'); ylabel('y')
```

