MAE 502, Fall 2015, HW2 Solutions

Prob 1

(a) The solution is

$$u(x,t) = 2 + \ln(1+t) + [e^{-\pi^2 t} + \frac{e^{-t} - e^{-\pi^2 t}}{\pi^2 - 1}]\cos(\pi x) + e^{-4\pi^2 t}\cos(2\pi x)$$

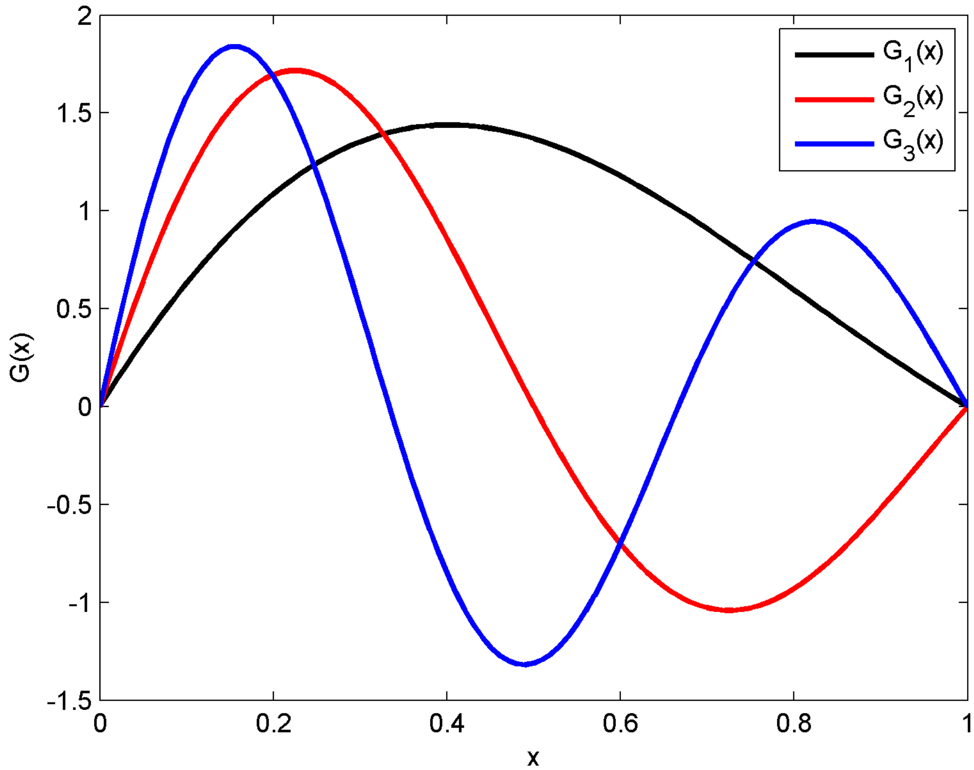
(b) The solution in Part (a) does not have a steady state because $\ln(1+t) \to \infty$ as $t \to \infty$. All other terms in the solution either stay finite or decay to zero as $t \to \infty$. If the term, $1/(1+t)$, in the r.h.s. of the PDE is replaced by $1/(1+t)^2$, the solution will be similar to that obtained in Part (a) except that "$2 + \ln(1+t)$" is replaced by "$3 - 1/(1+t)$". Since this term approaches a constant of 3 as $t \to \infty$, the new system has a steady state solution, $u(x,t) = 3$ as $t \to \infty$.

Prob 2

(a) The eigenvalues are $c_n = -1 - (n\pi)^2$, $n = 1, 2, 3$ ..., and the corresponding eigenfunctions are

$$G_n(x) = e^{-x}\sin(n\pi x)\ ,\ \ n = 1, 2, 3, \ldots$$

(b) Plot of the first 3 eigenfunctions (normalized as per the instruction in the handout):



(c) The integral,

$$\int_0^1 G_n(x)G_m(x)\,dx = \int_0^1 e^{-2x}\sin(n\pi x)\sin(m\pi x)\,dx \ \neq 0, \text{ when } n \neq m\ .$$

Therefore, the eigenfunctions of this system do not satisfy the orthogonality relation.

Prob 3

The solution is

$$u(x, y) = u_1(x, y) + u_2(x, y) \ ,$$

where

$$u_1(x, y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \sinh(\sqrt{2} n\pi y) \ ,$$

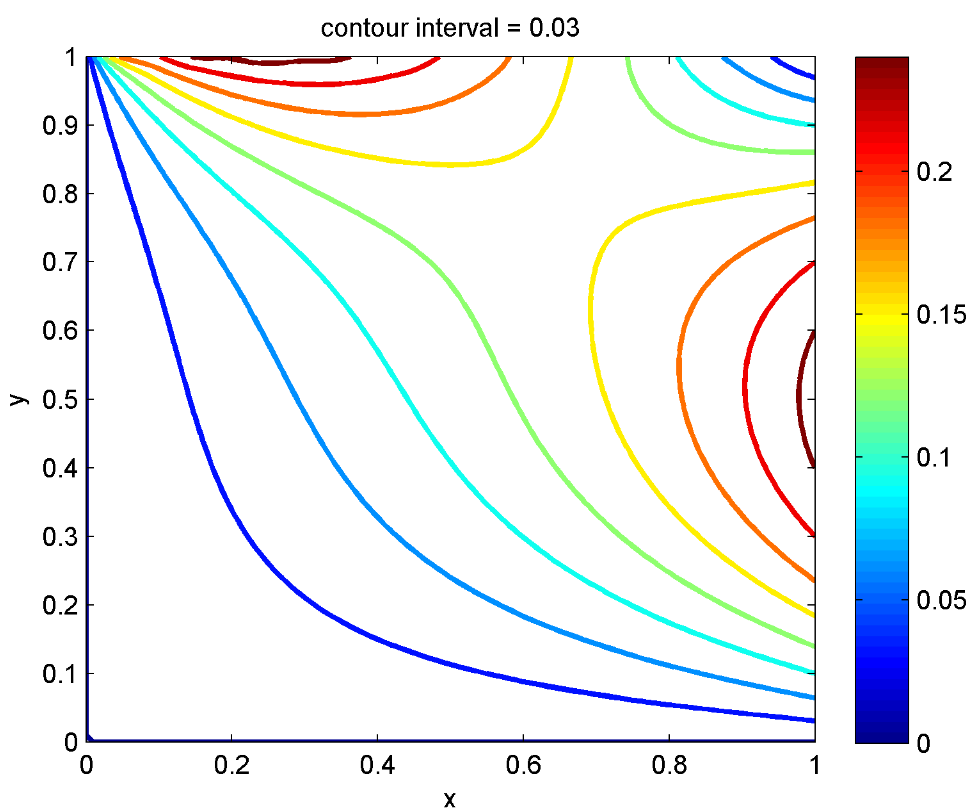
$$u_2(x, y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi x}{\sqrt{2}}\right) \sin(n\pi y) \ ,$$

and

$$a_n = \frac{2}{\sinh(\sqrt{2} n\pi)} \int_0^1 (\sqrt{x} - x) \sin(n\pi x) dx \ ,$$

$$b_n = \frac{2}{\sinh(n\pi/\sqrt{2})} \int_0^1 (y - y^2) \sin(n\pi y) dy \ .$$

Plot of the solution (with a truncation at 20 terms in both x- and y-direction):



$u(0.4, 0.6) = 0.0936$