MAE/MSE502 Fall 2015 Homework #4

You may find the following formula useful:

$$\int_0^{\infty} e^{-x^2} \cos(2bx)\, dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad .$$

In Prob 2, it can be used to evaluate the forward Fourier transform. (The inverse transform for that problem would rely on numerical integration.) For Prob 3 and 4, it can be applied twice in both forward and inverse Fourier transform to produce a close-form solution.

Prob 1 (3 points)
For $u(x,t)$ defined on the domain of $0 \leq x \leq 2\pi$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} - 0.1 \frac{\partial^4 u}{\partial x^4} \quad ,$$

with the boundary conditions (the first 4 conditions simply indicate that the system is periodic in the x-direction):

(i) $u(0, t) = u(2\pi, t)$, (ii) $u_x(0, t) = u_x(2\pi, t)$, (iii) $u_{xx}(0, t) = u_{xx}(2\pi, t)$, (iv) $u_{xxx}(0, t) = u_{xxx}(2\pi, t)$
(v) $u(x, 0) = \cos(x) + \sin(2x)$.

We expect a closed-form solution without any unevaluated integral(s) or summation of infinite series.

Prob 2 (4 points)
For $u(x,t)$ defined on the infinite domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + \exp[-(x^2 + t)] \quad ,$$

with the boundary conditions:

(i) $u(x, t)$ and its partial derivatives in x vanish as $x \to \pm\infty$
(ii) $u(x, 0) = \exp[-(x-3)^2]$.

It is acceptable to express the solution as an integral. Plot the solution as a function of x at $t = 0$, 0.3, and 2. Please collect all 3 curves in one plot.

Note: For this problem, numerical integration (e.g., by the trapezoidal method) might be needed to evaluate $u(x, t)$ for the plot. Since numerical integration cannot go all the way to ∞, one has to "truncate" the integral at a finite value of ω. This is analogous to truncating a Fourier series at a finite n. A useful way to determine where to truncate the integral is to plot, for a give t, $U(\omega, t)$ (the Fourier transform of $u(x, t)$) as a function of ω and observe how $U(\omega, t)$ decays with ω.

Prob 3 (1.5 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, use the method of Fourier transform to solve the PDE

$$\frac{\partial u}{\partial t} = (1+2t)\frac{\partial^2 u}{\partial x^2} \quad ,$$

with the boundary conditions:

(i) $u(x, t)$ and its partial derivatives in x vanish as $x \to \pm\infty$
(ii) $u(x,0) = \exp(-x^2)$.

To receive full credit, the final solution should have a closed-form expression of a *real* function which contains no unevaluated integral(s).

Prob 4 (1.5 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, use the method of Fourier transform to solve the PDE

$$\frac{\partial u}{\partial t} = 3\frac{\partial u}{\partial x} - \frac{u}{1+t} \quad ,$$

with the boundary conditions:

(i) $u(x, t)$ and its partial derivatives in x vanish as $x \to \pm\infty$
(ii) $u(x,0) = \exp(-x^2)$.

To receive full credit, the final solution should have a closed-form expression of a *real* function which contains no unevaluated integral(s). Plot the solution, $u(x,t)$, as a function of x at t = 0, 0.5, and 1. Please collect all 3 curves in a single plot.