MAE/MSE 502, Fall 2015 Homework #5

Prob 1 (4 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, find the solution of the PDE,

$$\frac{\partial u}{\partial t} + (1+u)\,\frac{\partial u}{\partial x} = 0 \quad ,$$

with the boundary condition,

$$u(x, 0) = \mathrm{P}(x) \ ,$$

where

$$\begin{aligned} \mathrm{P}(x) &= x && , \text{ if } 0 \leq x < 1 \\ &= (3-x)/2 && , \text{ if } 1 \leq x \leq 3 \\ &= 0 && , \text{ otherwise .} \end{aligned}$$

Plot the solution, $u(x,t)$, as a function of x at $t = 0$, 0.5, 1, and 1.5. (You may simply use the given P(x) to make the plot for $t = 0$.) Make a plot of selected characteristics in the x-t plane. Can finite-time blowup occur in the solution of this system? If so, what is the critical time, $t = t_c$, beyond which multiple solutions emerge?

Prob 2 (4 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, find the solution of the PDE,

$$\frac{\partial u}{\partial t} + 2u\,\frac{\partial u}{\partial x} = -u \quad ,$$

with the boundary condition,

$$u(x, 0) = \exp(-x^2) \ .$$

Plot the solution as a function of x at $t = 0$, 0.2, and 0.5. (You may simply use the given boundary condition at $t = 0$ to make the plot for $t = 0$.) We do not expect a closed-form solution for this problem. It suffices to express the solution as a procedure that can be used to systematically evaluate $u(x,t)$ with given x and t. The plot can be made with the assistance of a numerical procedure such as bisection or Newton's method.

Prob 3 (1.5 point)
For $u(x, y, t)$ defined on the domain of $-\infty < x < \infty$, $-\infty < y < \infty$, and $t \geq 0$, find the solution of the PDE,

$$\frac{\partial u}{\partial t} + x\,\frac{\partial u}{\partial x} + t\,\frac{\partial u}{\partial y} = 2u \quad ,$$

with the boundary condition,

$$u(x, y, 0) = \exp[-(x^2 + y^2)] \ .$$

Prob 4 (1.5 point)
For $u(x, t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, find the solution of the PDE,

$$(1+t)\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = tu \quad ,$$

with the boundary condition,

$$u(x, 0) = \exp(-x^2) \; .$$

Prob 5 (3 points)
Consider the following PDE for $u(x, t)$ defined on the infinite domain of $-\infty < x < \infty$ and $t \geq 0$,

$$\frac{\partial u}{\partial t} = -(3+t)\, u + Q(t) \quad ,$$

with the boundary condition,

$$u(x, 0) = \mathrm{P}(x).$$

Find the Green's function, $G(t, t')$, such that for any given $Q(t)$ and $\mathrm{P}(x)$ the solution of the system can be expressed as

$$u(x,t) = G(t,0)P(x) + \int_0^t G(t,t')Q(t')dt' \quad .$$