MAE/MSE 502, Spring 2015 Homework #1

1 point ≈ 1% of your total score for this class

Please submit hard copy of your work. Electronic submission will not be accepted. Please provide the print out of computer codes used in the work. The rules for collaboration on homework will be released separately. Please always follow the rules.

Prob. 1 (4 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ ,$$

with the boundary conditions (be aware that the first condition is imposed on the derivative of u),

(i) $u_x(0, t) = 0$, (ii) $u(1, t) = 0$, and (iii) $u(x,0) = -10\, x^3 + 9\, x^2 + 1$.

Plot the solution, $u(x,t)$, as a function of x at $t = 0$, 0.02, 0.05, 0.2, and 0.5. Please collect all five curves in a single plot. Hand-drawn figures are not acceptable. Please make sure the plot is clearly labeled.

See additional note in page 3 for a tip on using Matlab to evaluate integrals that arise from the application of an orthogonality relation. While some of those integrals can be evaluated analytically (by hand), it is perfectly fine to evaluate them numerically.

Note: If your solution is expressed as an infinite series, it is your job to determine the appropriate number of terms to keep in the series to ensure that the solution is accurate. As a useful measure, the solution at $t = 0$ should nearly match the given initial state in the 3rd boundary condition. If they do not match, either the solution is wrong or you have not retained enough terms in the series. This remark applies to all future homework problems that require the evaluation of an infinite series.

Prob. 2 (1.5 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \pi^2 u \ ,$$

with the boundary conditions,

(i) $u_x(0, t) = 0$, (ii) $u_x(1, t) = 0$, and (iii) $u(x,0) = 2 + 3\cos(\pi x) + 7\cos(4\pi x)$.

For this problem, we expect a closed-form analytic solution without any unevaluated integral or infinite series. A deduction will be assessed for any such items that are left untreated in the final answer. No need to make any plot for this problem.

Prob. 3 (1 point)
For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} + t\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u(1, t) = 0$, and (iii) $u(x, 0) = 5\sin(3\pi x)$.

For this problem, we expect a closed-form analytic solution without any unevaluated integral or infinite series. A deduction will be assessed for any such items that are left untreated in the final answer. No need to make any plot for this problem.

Prob. 4 (0.5 point)
For the heat transfer problem, the Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K\,\frac{\partial^2 u}{\partial \hat{x}^2} \;,\; 0 \le \hat{x} \le L \;\; (L \text{ is the length of the "metal rod", in meters) and } \hat{t} \ge 0\,, \quad (1)$$

where $\hat{t}$ and $\hat{x}$ are time in seconds and distance in meters, K is thermal diffusivity in m^2/s, and u is temperature. In our class, we usually consider the non-dimensionalized version of (1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;,\; 0 \le x \le 1 \;\text{ and }\; t \ge 0 \;, \qquad (2)$$

where x is related to $\hat{x}$ by $\hat{x} = L\,x$. In (2), time is also non-dimensionalized by $\hat{t} = T\,t$, where T is a certain dimensional time scale, and so on.

(a) In order to claim that the non-dimensionalized system, (1), is equivalent to its dimensional counterpart, (2), the three parameters K, L, and T must satisfy a unique relation. First, find out what this relation is. (For the discussion in part (b) & (c), it is useful to write the relation as $T = f(K, L)$.)

(b) Suppose that the non-dimensionalized Heat equation, (2), is used to model the real world problem of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem?

(c) Same as (b), but suppose that (2) describes heat transfer along a wooden stick that is 0.3 meter long and made of pine wood ($K \approx 10^{-7}$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to? (We consider $t = 0.01$ because it is about the time when a significant redistribution of temperature begins to take place in the scenario described in Part (d).)

(d) Are the time scales you obtained in (b) and (c) consistent with daily experience? For instance, one can use a long wooden spoon to continuously stir a boiling pot of soup without getting one's hand burned. In contrast, the same practice would make one very uncomfortable if the spoon is made entirely of copper. Note that the time scale for cooking a pot of soup is about 10 minutes. The length of a big wooden spoon is about a foot, or 0.3 meter.

Prob. 5 (1 point)
(a) Consider the following PDE which looks similar to the 1-D heat equation but has a negative sign in the right hand side,

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} \quad .$$

Given the boundary conditions, (i) $u(0,t) = 0$, (ii) $u(1,t) = 0$, and (iii) $u(x,0) = P(x)$ (where $P(x)$ is a well behaved function and is not identically zero), discuss how the solution of this equation would differ qualitatively from the solution of the Heat equation given the same boundary conditions. Discuss the behavior of the solution as $t \to \infty$.

(b) Consider the 1-D heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad ,$$

with the boundary conditions: (I) $u(0,t) = 2$, (II) $u_x(1,t) = 0$, and (III) $u(x,0) = 2 + 5\sin(1.5\pi x)$. Does this system have a steady state solution? If yes, find it. If no, explain why.

Additional note: How to use Matlab to numerically evaluate an integral

The simplest Matlab function for this purpose is perhaps **trapz**. It uses the trapezoidal method to evaluate an integral. For example, to numerically evaluate

$$I = \int_0^1 \sin(x)\, dx$$

with $\Delta x = 0.01$, we first construct the discretized arrays of the coordinate points and the values of the integrand at those points. We then call **trapz** with those two arrays as the input to complete the integration. The Matlab code is very simple:

```
x = [0:0.01:1];
y = sin(x);
Integ = trapz(x,y)
```

One can readily verify the outcome with the analytic result of $I = 1 - \cos(1)$.