# MAE/MSE502, Spring 2015 Homework #3

**Prob 1 (3 points)**
Consider the eigenvalue problem,

$$\frac{d^2 G}{d x^2} = c\ G \quad , \ G'(0) = 5 \ , G(1) = 3 \quad \text{(note that the first b.c. is imposed on the derivative of } G\text{)}$$

**(a)** Determine the eigenvalues and the corresponding eigenfunctions of this problem.
Do consider all three possibilities with $c > 0$, $c = 0$, and $c < 0$. Are the eigenvalues discrete? For example, if the boundary conditions are replaced by the familiar $G(0) = 0$ and $G(1) = 0$, we would have $c = c_n = -n^2 \pi^2$ ($n$ is an integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation when the eigenvalues are not discrete is if all values within an interval, $A \le c \le B$, are valid eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

**(b)** Plot the eigenfunctions, $G_C(x)$, associated with the eigenvalues c = –50, –10, –5, 0, 5, 10, and 50. (You will find in Part (a) that all those values are indeed valid eigenvalues.) Please collect all 7 curves in a single plot. Note that $G(x)$ is defined only on the interval of $0 \le x \le 1$. Your plot should cover only that interval.

**(c)** Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^1 G_p(x) G_q(x) dx = 0, \ \ if\ p \neq q \quad ,$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions that correspond to two distinctive eigenvalues $p$ and $q$? Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of $G_p(x)G_q(x)$ and show that it leads to a non-zero value even when $p \neq q$. One such counterexample would suffice to prove that the orthogonality relation does not hold. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all pairs of $p$ and $q$.

**(d)** If $G_p(x)$ is an eigenfunction corresponding to an eigenvalue, $c = p$, would $AG_p(x)$ (where $A$ is an arbitrary constant; $A \neq 1$) also be an eigenfunction? Provide a brief explanation to support your yes/no answer.

**Prob 2 (5 points)**

[Please see the additional note in pp. 3-5 for useful information related to this problem.]

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, consider the Heat equation with a non-constant thermal diffusivity and non-constant internal dissipation,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K(x)\frac{\partial u}{\partial x}\right] - D(x)u \quad ,$$

with the boundary conditions,

(i) $u(0, t) = 0$ , (ii) $u(1, t) = 0$ , and (iii) $u(x,0) = (x - x^2)\, e^{2x}$ .

With a non-constant $K(x)$, the approach of separation of variables and eigenfunction expansion will still work because the ODE and b.c.'s in the $x$-direction obtained by separation of variables will form a Sturm-Liouville system. You are required to solve the problem using the following approach (*cf.* Sec 5.4): (i) Solve the Sturm-Liouville system to obtain the eigenvalues and eigenfunctions, (ii) Expand $u(x,t)$ in terms of the eigenfunctions in $x$ (and their counterparts in the $t$-direction), (iii) Find the expansion coefficients by applying the orthogonality relation of the eigenfunctions. With a non-constant $K(x)$, the eigenfunctions will no longer be sine and cosine. Although one might still be able to solve them analytically (which is allowed), it is recommended that the eigenvalues and eigenfunctions be obtained numerically by the finite difference method. Additional instructions on the technical detail will be given in class.

(a) Using the approach of separation of variables and eigenfunction expansion, solve the case with $K(x) = 4x^2 + x + 0.05$, $D(x) = 0$. Plot the solution as a function of $x$ at $t = 0$, 0.01, 0.05, and 0.15. To receive full credit, you must also provide:

(I) The five eigenvalues with the <u>smallest absolute values</u>.
(II) A plot of the five eigenfunctions corresponding to the five eigenvalues given in (I). Please collect all five curves in one figure.

The eigenvalues and eigenfunctions in (I) and (II) refer to those obtained by solving the Sturm-Liouville system in the $x$-direction. The eigenfunctions will be functions of $x$ only.

(b) Repeat (a) but solve the case with $K(x) = 4x^2 + x + 0.05$, $D(x) = 5e^{2x}$. Plot the solution as a function of $x$ at $t = 0$, 0.01, 0.05, and 0.15, and <u>provide the additional information described in (I) and (II) in Part (a)</u>. Be aware that having the non-zero $D(x)$ will affect the eigenvalues and eigenfunctions.

Note: Although you are asked to make the plots for only five eigenfunctions, more than five should be included in the eigenfunction expansion in order to obtain an accurate solution for $u(x,t)$. As a check of the accuracy of your solution, the solution at $t = 0$ should closely match the "initial state" given in the 3rd b.c.

**Useful background for Prob 2**

By separation of variables, the original PDE in Prob 2 can be converted into an eigenvalue problem in the $x$-direction and a simple 1st order ODE in the $t$-direction. The former is of the standard Sturm-Liouville form, which guarantees that its eigenvalues are discrete and its eigenfunctions are mutually orthogonal. Sturm-Liouville theorem does not tell us how to actually obtain the eigenvalues and eigenfunctions. To do so, there are two approaches:

(1) Find the general solution of the 2nd order ODE in the $x$-direction analytically, use the boundary conditions to determine the eigenvalues, then insert the eigenvalues into the general solutions to determine the eigenfunctions. For this problem, this is the more challenging approach to take.

(2) Instead, our recommendation is to solve the eigenvalues and eigenfunctions by the finite difference method. To illustrate how it works, let's consider a simpler problem which we know how to solve analytically:

$$G'' = c\,G \;,\quad G(0) = 0,\; G(1) = 0.$$

Pretending that we do not know the analytic solution (which is $c_n = -n^2\pi^2$, $G_n(x) = \sin(n\pi x)$, $n = 1, 2, 3, ...$), by finite difference approximation (using a 2nd order central F.D. scheme for the 2nd derivative) we have

$$G''(x_i) \approx \; [G(x_{i-1}) - 2\,G(x_i) + G(x_{i+1})]/(\Delta x^2)\;,$$

where $x_i \equiv (i-1)\Delta x$, $\Delta x \equiv 1/(N-1)$, and N is the total number of grid points including the end points of the interval. We will use the shorthand, $G_i \equiv G(x_i)$. The boundary conditions now become

$$G_1 = 0 \;\;, G_N = 0.$$

Therefore, the eigenvalue problem can be rewritten as

$$G_{i-1} - 2\,G_i + G_{i+1} = c\,G_i\;,\quad i = 2, 3, ..., N-1$$

This is essentially a matrix eigenvalue problem which can be written as

$$\frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & \mathbf{0} & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \begin{pmatrix} G_2 \\ G_3 \\ G_4 \\ \vdots \\ G_{N-2} \\ G_{N-1} \end{pmatrix} = c \begin{pmatrix} G_2 \\ G_3 \\ G_4 \\ \vdots \\ G_{N-2} \\ G_{N-1} \end{pmatrix}$$

The matrix eigenvalue problem can be solved using Matlab. The following is an example of the Matlab code for the end-to-end solution of the above eigenvalue problem. Please consult the online documentation for the **eig** function in Matlab (http://www.mathworks.com/help/matlab/ref/eig.html) to understand its proper usage. Note that the eigenvectors so obtained are essentially our eigenfunctions discretized at $x_i$. No further distinction will be made between the two. In this example, the first 3 eigenvalues (with the smallest absolute values) from the numerical solution are –9.8688 , –39.4654, and –88.7607, which very closely match the analytic results of $-\pi^2$, $-4\pi^2$, and $-9\pi^2$. The eigenfunctions (see plot in the next page) corresponding to those eigenvalues also closely match their analytic counterparts of $\sin(\pi x)$, $\sin(2\pi x)$, and $\sin(3\pi x)$.

For the more complicated eigenvalue problem in the homework, the finite difference approximation needs to be applied to all terms in the ODE that involve differentiation in $x$. For the first derivative, the 2nd order central F.D. scheme is recommended.

Important: Matlab does not automatically sort the eigenvalues in ascending or descending order. This will cause some inconvenience if we wish to use only the first M (< N) eigenvectors (corresponding to the first M eigenvalues with the smallest absolute values) to do the expansion for the solution of the PDE. As a quick fix, the following Matlab code has included a segment for sorting the eigenvalues and reordering the corresponding eigenvectors. Please read the code carefully before adopting it for your own use.

```
clear
% -- written by HP Huang --
dx = 0.01; x = [0:dx:1]; N = length(x); v = zeros(N,N-2);
c = zeros(N-2); V = zeros(N-2,N-2); D = zeros(N-2,N-2); S = zeros(N-2,N-2);
cs = zeros(N-2); vs = zeros(N,N-2);
% -- construct the main matrix for the eigenvalue problem
S(1,1) = -2; S(1,2) = 1; S(N-2,N-2) = -2; S(N-2,N-3) = 1;
for i = 2:N-3
        S(i,i) = -2; S(i,i-1) = 1; S(i,i+1) = 1;
end
S = S/(dx^2);
% -- solve the eigenvalue problem
[V D] = eig(S);
for i = 1:N-2
    c(i) = D(i,i);
    v(:,i) = [0 V(:,i)' 0];
end
% -- the following section is for sorting the eigenvectors in descending
%    order of the absolute value of their corresponding eigenvalues
for i = 1:N-2
    crank(i) = 1;
    for j = 1:N-2
        if abs(c(i)) > abs(c(j))
            crank(i) = crank(i)+1;
        end
    end
end
for i = 1:N-2
    cs(crank(i)) = c(i);
    vs(:,crank(i)) = v(:,i);
end
% -- output the values of the first 3 eigenvalues and plot the
%    corresponding eigenfunctions
cs(1:3)
plot(x,vs(:,1),'k-',x,vs(:,2),'k--',x,vs(:,3),'k-.')
xlabel('x'),ylabel('G(x)')
legend('1st eigf','2nd eigf','3rd eigf','Location','SouthEast')
```

The first three eigenfunctions produce by the Matlab code: