MAE/MSE 502, Spring 2015 Homework #6

Prob 1
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, consider the PDE,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = 0$$

with the boundary condition,

$u(x, 0) = \mathrm{P}(x)$.

(a) (1.5 points) Solve $u(x,t)$ for the case with

$$\begin{aligned} \mathrm{P}(x) &= 1\,, && \text{if } x < 0 \\ &= 1 + x\,, && \text{if } 0 \leq x \leq 1 \\ &= 2\,, && \text{if } x > 1\,. \end{aligned}$$

Plot $u(x,t)$ as a function of x at $t = 0$, 0.5, and 1. (For $t = 0$, you can just use the given $\mathrm{P}(x)$ for the plot.) In addition, make a plot of selected characteristics in the x-t plane and use it to discuss the properties of the solution.

(b) (2 points) Solve $u(x,t)$ for the case with

$$\begin{aligned} \mathrm{P}(x) &= x - x^2\,, && \text{if } 0 \leq x \leq 1 \\ &= 0, && \text{otherwise} \end{aligned}$$

Plot $u(x,t)$ as a function of x at $t = 0$, 0.5, and 1. (For $t = 0$, you can just use the given $\mathrm{P}(x)$ for the plot.) In addition, make a plot of selected characteristics in the x-t plane and use it to discuss the properties of the solution. Would the solution become multiple-valued within a finite time? If so, what is the critical value of t beyond which $u(x,t)$ becomes multiple-valued?

Prob 2 (0.5 point)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE,

$$\frac{\partial u}{\partial t} + x\,t\frac{\partial u}{\partial x} = x\,t\,u \quad ,$$

with the boundary condition,

$u(x, 0) = \exp(-x^2)$.

Prob 3 (0.5 point)
For $u(x,y,t)$ defined on the domain of $-\infty < x < \infty$, $-\infty < y < \infty$, and $t \geq 0$, solve the PDE,

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = t \quad ,$$

with the boundary condition,

$$u(x, y, 0) = \exp[-(x^2+y^2)] \ .$$

Prob 4 (1.5 point)
Consider the following PDE for $u(x, t)$ defined on the infinite domain of $-\infty < x < \infty$ and $t \geq 0$,

$$\frac{\partial u}{\partial t} = -3tu + Q(t) \quad ,$$

with the boundary condition,

$$u(x, 0) = \mathrm{P}(x).$$

Find the Green's function, $G(t, t')$, such that for any given $Q(t)$ and $\mathrm{P}(x)$ the solution of the system can be expressed as

$$u(x,t) = G(t,0)\,P(x) + \int_0^t G(t,t')Q(t')\,dt' \quad .$$