MAE/MSE 502, Spring 2016 Homework #1

1 point ≈ 1% of your total score for this class

Hard copy of your work is due *before class* on the due date. The rules for collaboration on homework will be released separately.

Prob. 1 (3 points)

For $u(x, t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the modified Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 3u \quad ,$$

with the boundary conditions (be aware that the first condition is imposed on the derivative of u),

(i) $u_x(0, t) = 0$, (ii) $u(1, t) = 0$, and   (iii) $u(x, 0) = -3x^4 + 2x^3 + 1$.

Plot the solution, $u(x,t)$, as a function of x at $t = 0$, 0.03, 0.1, and 0.3. Please collect all five curves in a single plot.

Note: We expect the solution to be expressed as an infinite series. It is part of your job to determine the appropriate number of terms to keep in the series to ensure that the solution is accurate. As a useful measure, the solution at $t = 0$ should nearly match the given initial state in the 3rd boundary condition. If they do not match, either the solution is wrong or more terms need to be retained in the series. This remark applies to all future homework problems that require the evaluation of an infinite series. See additional note in the last page for an example of using Matlab to numerically evaluate integrals. It might help the computation of the expansion coefficients.

Prob. 2 (3 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, consider the PDE,

$$\frac{\partial u}{\partial t} = 3\,\frac{\partial^2 u}{\partial x^2} - u \quad ,$$

with the boundary conditions,

(i) $u_x(0, t) = 2$, (ii) $u_x(1, t) = 4$, and   (iii) $u(x,0) = x^2 + 2x$.

The "total energy" of the system is defined as

$$E(t) \equiv \int_0^1 u(x,t)\, dx \quad .$$

(a) Find the exact (analytic) expression of $E(t)$. Make a plot of $E(t)$ as a function of t over the range of $0 \le t \le 2$. What is the value of $E(t)$ as $t \to \infty$?

(b) Does a steady state exist for this system? If no, explain why. If yes, find the steady state solution, $u_S(x)$, and plot it along with the given initial state, $u(x, 0)$, in a single figure.

(c) Verify that your answer to Part (b) is consistent with the answer to Part (a). Specifically, if a steady state does not exist, then $E(t)$ should blow up (i.e., it approaches ∞ or $-\infty$) as $t \rightarrow \infty$. If a steady state exists, then the integral

$$\int_0^1 u_S(x)\,dx$$

should equal the value of $E(t)$ as $t \rightarrow \infty$.

(You may or may not need to find the full solution, $u(x, t)$, for all x and t, in order to answer the questions in Part (a)-(c).)

Prob. 3 (1.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \sin(t) \quad ,$$

with the boundary conditions,

(i) $u_x(0, t) = 0$, (ii) $u_x(1, t) = 0$, and (iii) $u(x,0) = 3 + 4\cos(\pi x)$.

We expect a closed-form analytic solution that consists of only a finite number of terms and no unevaluated integrals. A deduction will be assessed on any unevaluated integral or undetermined coefficient that is left in the final answer.

Prob. 4 (0.5 point)
For the heat transfer problem, the Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K \frac{\partial^2 u}{\partial \hat{x}^2} \quad , \; 0 \le \hat{x} \le L \;\; (L \text{ is the length of the "metal rod", in meters) and } \hat{t} \ge 0 , \quad (1)$$

where $\hat{t}$ and $\hat{x}$ are time in seconds and distance in meters, K is thermal diffusivity in m^2/s, and u is temperature. In our class, we usually consider the non-dimensionalized version of (1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , \; 0 \le x \le 1 \;\text{ and }\; t \ge 0 \;, \qquad (2)$$

where x is related to $\hat{x}$ by $\hat{x} = L\,x$. In (2), time is also non-dimensionalized by $\hat{t} = T\,t$, where T is a certain dimensional time scale, and so on.

(a) In order to claim that the non-dimensionalized system, (2), is equivalent to its dimensional counterpart, (1), the three parameters K, L, and T must satisfy a unique relation. First, find out what this relation is. (For the discussion in Part (b) & (c), it is useful to write the relation as $T = f(K, L)$.)

(b) Suppose that the non-dimensionalized Heat equation, (2), is used to model the real world problem of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem?

(c) Same as (b), but suppose that (2) describes heat transfer along a wooden stick that is 0.3 meter long and made of pine wood ($K \approx 10^{-7}$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to? (We consider $t = 0.01$ because it is about the time when a significant redistribution of temperature begins to take place in the scenario described in Part (d).)

(d) Are the time scales you obtained in (b) and (c) consistent with daily experience? For instance, one can use a long wooden spoon to continuously stir a boiling pot of soup without getting one's hand burned. In contrast, the same practice would make one very uncomfortable if the spoon is made entirely of copper. Note that the time scale for cooking a pot of soup is about 10 minutes. The length of a big wooden spoon is about a foot, or 0.3 meter.

Additional note: How to use Matlab to numerically evaluate an integral

One of the simplest Matlab functions for this purpose is **trapz**. It uses the trapezoidal method to evaluate an integral. For example, to numerically evaluate

$$I = \int_0^1 \sin(x)\, dx \quad ,$$

with $\Delta x = 0.01$, we first construct the discretized arrays of the coordinate points and the values of the integrand at those points. We then call **trapz** with those two arrays as the input to complete the integration. The Matlab code is very simple:

```
x = [0:0.01:1];
y = sin(x);
Integ = trapz(x,y)
```

One can readily verify the outcome with the analytic result of $I = 1 - \cos(1)$.