

MAE/MSE502 Spring 2016 Homework #2

Hard copy of report is due before class on the due date. Please submit printout of Matlab code (if any).

Prob 1 (4 points)

For $u(x, y)$ defined on the rectangular domain of $0 \leq x \leq 2$ and $0 \leq y \leq 1$, solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$

with the boundary conditions,

$$(i) u(0, y) = 0 \quad , \quad (ii) u(2, y) = 8y - 8y^2 \quad , \quad (iii) u(x, 0) = 4x^3 - 13x^2 + 10x \quad , \quad (iv) u(x, 1) = 0.$$

Make a contour plot of your solution in the x - y plane. Based on the solution, evaluate $u(x, y)$ at $x = 1.2, y = 0.6$. (Major deduction if this value is not calculated.) See **Additional Note** for an example of Matlab code for making a contour plot. For this problem, the recommended contour interval is around 0.2.

Prob 2 (2 points)

For $u(x, y)$ defined on the square domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, solve the PDE,

$$\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial y^2} + \pi^2 u = 0 \quad ,$$

with the boundary conditions,

$$(i) u_x(0, y) = 0 \quad , \quad (ii) u_x(1, y) = 0 \quad , \quad (iii) u(x, 0) = 5 \cos(2\pi x) \quad , \quad (iv) u(x, 1) = 2 + \cos(\pi x).$$

For this problem, we expect a closed-form analytic solution with a finite number of terms and with no unevaluated integrals. A deduction will be assessed on any unevaluated integral that is left in the final answer.

Prob 3 (3 points)

For $u(x, t)$ defined on the domain of $0 \leq x \leq 4$ and $t \geq 0$, consider the 1-D Wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad ,$$

with the boundary conditions,

$$(i) u(0, t) = 0 \quad , \quad (ii) u(4, t) = 0 \quad , \quad (iii) u(x, 0) = P(x) \quad , \quad (iv) u_t(x, 0) = 0 \quad .$$

(a) Solve the system with $P(x)$ given as

$$P(x) = -x/3, \text{ if } 0 \leq x \leq 3 \\ = x - 4, \text{ if } 3 < x \leq 4,$$

and plot the solution as a function of x at $t = 0, 1.2, 2, 2.8, 4,$ and 7.2 . Please collect all 6 curves in one plot.

(b) Solve the system with $P(x)$ given as

$$P(x) = \sum_{n=76}^{84} (1 - 0.2 |n - 80|) \sin\left(\frac{n\pi x}{4}\right),$$

and plot the solution as a function of x at $t = 0, 1, 2,$ and 4 . For this part only, it is recommended that the four curves be plotted separately, for example in four panels using the "subplot" function in Matlab.

Prob 4 (2 points)

For $G(x)$ defined over the interval, $0 \leq x \leq 1$, consider the eigenvalue problem,

$$\frac{d^2 G}{dx^2} + 4 \frac{dG}{dx} = c G, \quad G(0) = 0, \quad G(1) = 0.$$

(a) Determine the eigenvalues and the corresponding eigenfunctions of this problem.

(b) Plot the three eigenfunctions associated with the three eigenvalues with the smallest absolute values. Since the amplitude of an eigenfunction can be arbitrary for this problem, it is recommended that in the plot each eigenfunction be normalized such that

$$\int_0^1 [G(x)]^2 dx = 1.$$

(c) Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^1 G_p(x) G_q(x) dx = 0, \quad \text{if } p \neq q,$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions associated with two distinctive eigenvalues p and q ? Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of $G_p(x)G_q(x)$ and show that it leads to a non-zero value even when $p \neq q$. One such counterexample would suffice to prove that the orthogonality relation does not hold. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all pairs of p and q .

Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x)\exp(-2y)$ over the square domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, using the contour levels of $(-0.9, -0.7, -0.5, -0.3, -0.1, -0.05, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9)$. The contours for $u = -0.7, -0.3, 0.3$, and 0.7 are labeled. It is essential to provide the coordinates of the grid ($x2d$ and $y2d$ in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y . A black-and-white contour plot is acceptable as long as the contours are properly labeled.

```
clear
x = [0:0.01:1]; y = [0:0.01:1];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = sin(2*pi*x(i))*exp(-2*y(j));
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
[C,h] = contour(x2d,y2d,u, [-0.9:0.2:-0.1 -0.05 0.05 0.1:0.2:0.9]);
clabel(C,h, [-0.7 -0.3 0.3 0.7])
xlabel('x'); ylabel('y')
```

