MAE/MSE 502, Spring 2016, HW2 Solutions

Prob 1

$$u(x, y) = u_1(x, y) + u_2(x, y) \quad ,$$

where

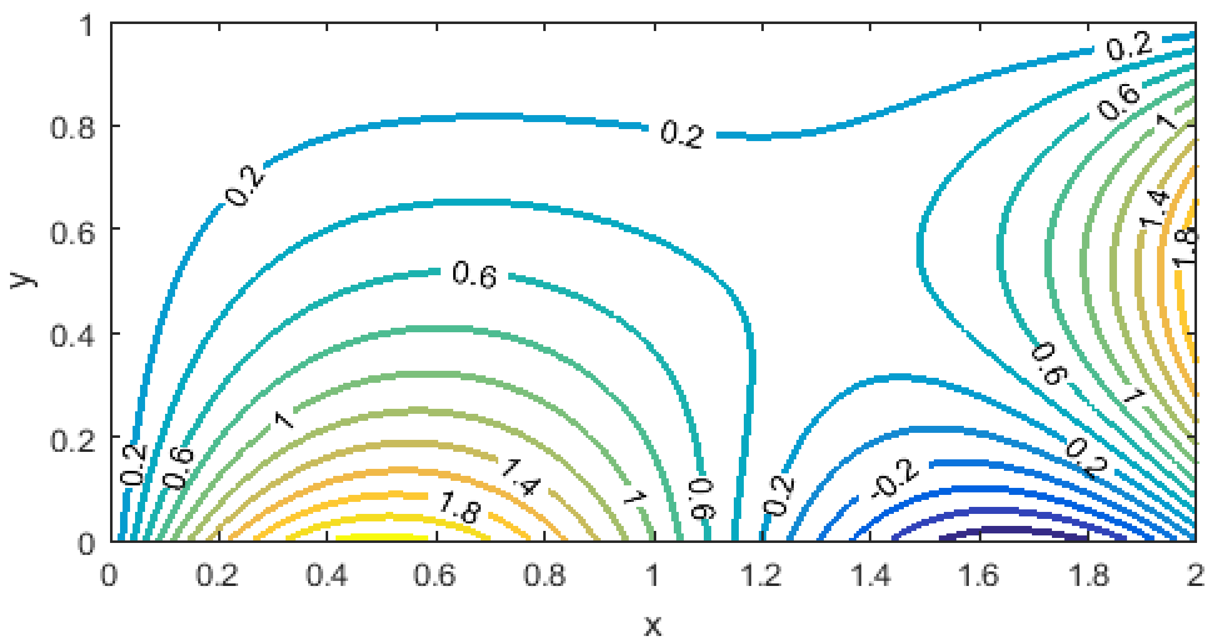
$$u_1(x, y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi x) \sin(n\pi y) \quad ,$$

$$u_2(x, y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \sinh\left(\frac{n\pi(1-y)}{2}\right) \quad ,$$

$$a_n = \frac{2}{\sinh(2n\pi)} \int_0^1 (8y - 8y^2) \sin(n\pi y)\, dy \quad ,$$

$$b_n = \frac{1}{\sinh\left(\frac{n\pi}{2}\right)} \int_0^2 (4x^3 - 13x^2 + 10x) \sin\left(\frac{n\pi x}{2}\right) dx \quad .$$

Plot of the solution (with the summation of each infinite series truncated, inclusively, at $n = 15$):



$u(1.2, 0.6) = 0.3204$

Prob 2

$$u(x,y) = 2\sin\left(\frac{\pi}{2}y\right) + y\cos(\pi x) + 5\left[\frac{\sinh\left(\frac{\sqrt{3}\pi}{2}(1-y)\right)}{\sinh\left(\frac{\sqrt{3}\pi}{2}\right)}\right]\cos(2\pi x)$$

Prob 3

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{n\pi t}{4}\right) \quad .$$

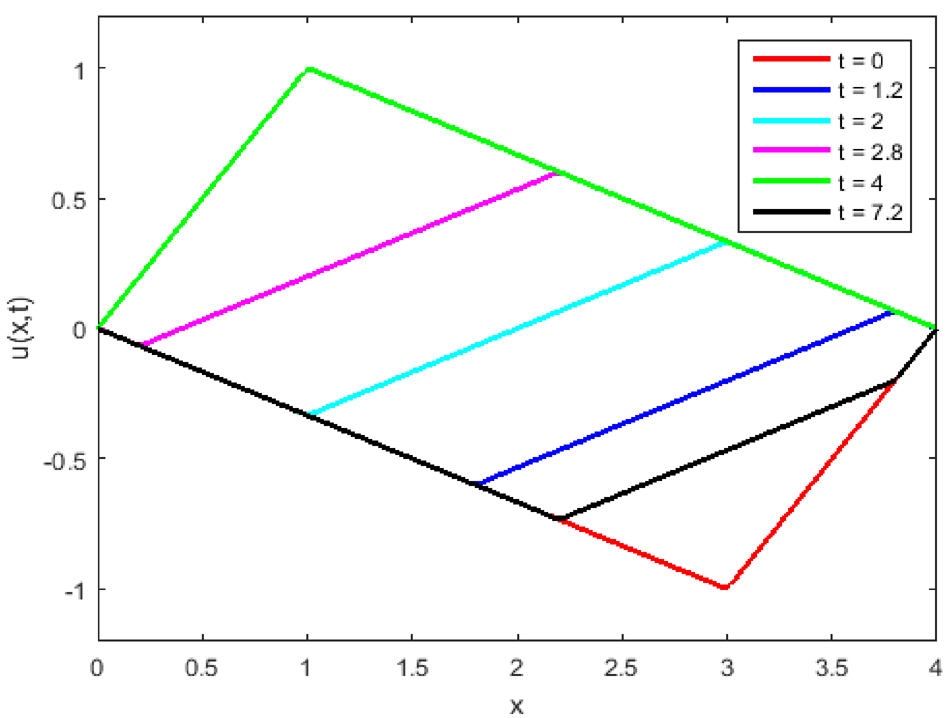
For Part (a),

$$a_n = \frac{1}{2}\left[\int_0^3 -\frac{x}{3}\sin\left(\frac{n\pi x}{4}\right)dx + \int_3^4 (x-4)\sin\left(\frac{n\pi x}{4}\right)dx\right] \quad .$$

For Part (b),

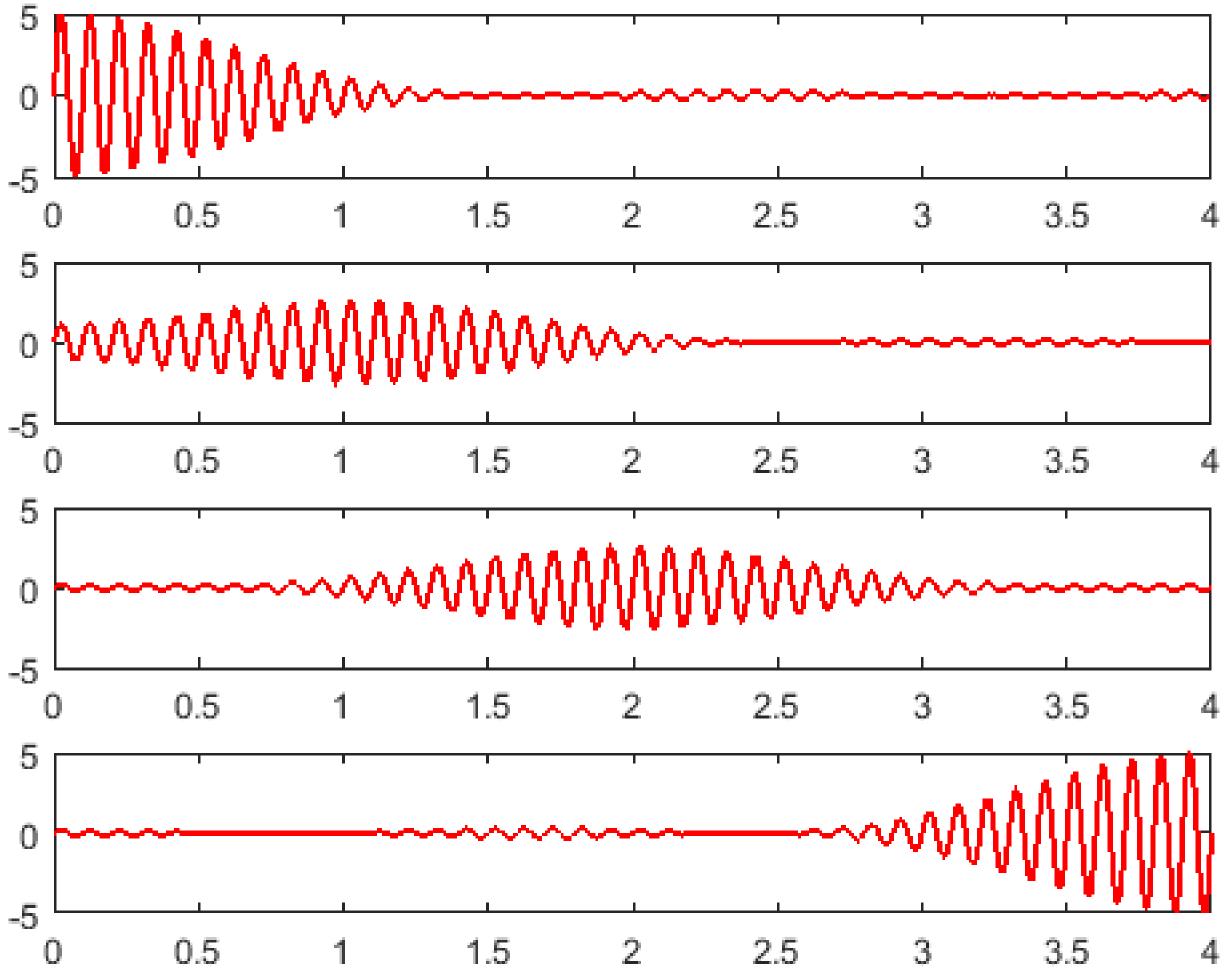
$$\begin{aligned} a_n &= 1-0.2\,|n-80| && \text{, if } 76 \le n \le 80 \\ &= 0 && \text{, otherwise .} \end{aligned}$$

Plot for Part (a):



Plot for Part (b):

Top to bottom: t = 0, 1, 2, and 4. Abscissa is x coordinate.



Prob 4

(a) The eigenvalues are $c_n = -4 - (n\pi)^2$. The corresponding eigenfunctions are

$$G_n(x) = e^{-2x} \sin(n\pi x) .$$

(b) The eigenfunctions can be normalized as

$$G_n(x) = \sqrt{\frac{8[(n\pi)^2 + 4]}{(n\pi)^2 (1 - e^{-4})}}\, e^{-2x} \sin(n\pi x) \quad ,$$

which will satisfy the normalization condition suggested in the handout. See plot of the first three $G_n(x)$ in the next page.

(c) The eigenfunctions do not satisfy the orthogonality relation.

Plot for Prob 4:

The first 3 normalized eigenfunctions are in red, blue, and gray.