MAE/MSE 502, Spring 2016, Homework #3

Please submit the printout of computer code(s) used in your work.

Prob. 1 (3 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} \quad ,$$

with the boundary conditions (the first four simply indicate that the system is periodic in the x direction),

(i) $u(0, t) = u(2\pi, t)$
(ii) $u_x(0, t) = u_x(2\pi, t)$
(iii) $u_{xx}(0, t) = u_{xx}(2\pi, t)$
(iv) $u_{xxx}(0, t) = u_{xxx}(2\pi, t)$
(v) $u(x, 0) = 3$
(vi) $u_t(x, 0) = 5 + \cos(x) + \sin(2x)$.

For this problem, we expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals.

Prob. 2 (4 points)

(a) For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} + E\,\frac{\partial^3 u}{\partial x^3} \quad ,$$

with the boundary conditions (the first three simply indicate that the system is periodic in the x direction)

(i) $u(0, t) = u(2\pi, t)$
(ii) $u_x(0, t) = u_x(2\pi, t)$
(iii) $u_{xx}(0, t) = u_{xx}(2\pi, t)$
(v) $u(x, 0) = \exp([1 - \cos(x)]^2)$,

with $D = 0$ and $E = 1$. Plot the solution, $u(x,t)$, as a function of x at $t = 0$, 1, and 2. Please collect all three curves in one plot.

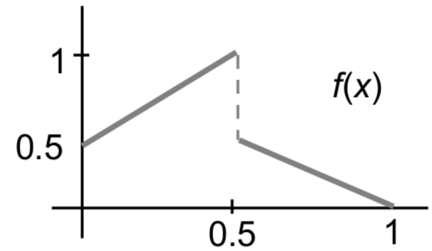
(b) Repeat **(a)** but for the case with $D = 0.2$ and $E = 1$.

(c) Repeat **(a)** but for the case with $D = 0$ and $E = -1$.

(d) Briefly discuss how the values (and signs) of D and E control the qualitative behavior of the solution of the system.

We expect the solution to be expressed as an infinite series. See relevant remarks below HW1-Prob 1 concerning the computation of the series.

Prob. 3 (3 points)
Consider the function defined on the interval of $0 \le x \le 1$ as (see sketch at right)



$$f(x) = 0.5 + x\,, \quad 0 \le x \le 0.5$$
$$\quad\;\; = 1 - x\,, \quad 0.5 < x \le 1\,.$$

Denote the Fourier Cosine series expansion of $f(x)$ as

$$F_C(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x) \quad ,$$

and Fourier Sine series expansion of $f(x)$ as

$$F_S(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad .$$

(a) Plot the original $f(x)$ and its Fourier Cosine series representation, $F_C(x)$, truncated (inclusively) at $n = 5$, 10, and 30. Please collect all four curves in a single plot.

(b) Define $S(\mathrm{N})$ as the value of $F_C(0.5)$ (i.e., the value of the Fourier Cosine series at $x = 0.5$) truncated (inclusively) at $n = \mathrm{N}$, plot $S(\mathrm{N})$ as a function of N over the range of $1 \le \mathrm{N} \le 100$. What value does $S(\mathrm{N})$ converge to at large N?

(c) Repeat (a) but now plot the original function $f(x)$ and its Fourier Sine series representation, $F_S(x)$. Compared to the Fourier Cosine series in (a), is Fourier Sine series a better or worse representation of $f(x)$? Briefly explain why.

Prob. 4 (1 point)
(a) Given the following function defined on the semi-infinite interval, $0 \le x < \infty$,

$$f(x) = 1 \;,\; \text{if } 0 \le x \le 1, \qquad \text{Eq. (1)}$$
$$\quad = 0 \;,\; \text{if } x > 1\,,$$

determine the Fourier Sine transform of $f(x)$, $F(\omega)$, that satisfies

$$f(x) = \int_0^{\infty} F(\omega) \sin(\omega x)\, d\omega \quad .$$

Plot $F(\omega)$ as a function of ω for the range of $0 \le \omega \le 30$.

(b) If the $f(x)$ in Eq. (1) is instead defined on a finite interval, $0 \le x \le L$ (but otherwise retaining its definition in Eq. (1), i.e., $f(x) = 0$ if $1 < x \le L$), find the coefficients, a_n, for the Fourier Sine series of $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad .$$

Plot a_n as a function of n for the following cases: (i) For $L = 2$, plot a_n over the range of $1 \leq n < 60/\pi$. (ii) For $L = 5$, plot a_n for $1 \leq n < 150/\pi$. (iii) For $L = 100$, plot a_n for $1 \leq n < 3000/\pi$. Compare these plots with the plot of $F(\omega)$ in Part (a). Discuss your results.

(Note: This exercise illustrates the correspondence between Fourier series and Fourier integral. When making the plots, be aware that the "n" in Part (b) is an integer while the "ω" in Part (a) can be any real number.)