MAE/MSE 502, Spring 2016 Homework #5

Prob 1 (3 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, find the solution of the PDE,

$$\frac{\partial u}{\partial t} + (u+1)^2 \frac{\partial u}{\partial x} = 0 \quad ,$$

with the boundary condition,

$$u(x, 0) = \mathrm{P}(x) \ ,$$

where

$$\begin{aligned} \mathrm{P}(x) &= 1 &&, \text{ if } x < 0 \\ &= 1 + x &&, \text{ if } 0 \leq x \leq 1 \\ &= 2 &&, \text{ if } x > 1 \end{aligned}$$

Plot the solution, $u(x,t)$, as a function of x at t = 0, 0.15, and 0.3. In addition, plot the characteristics in the x-t plane. Can finite-time blow-up occur for this system? Explain why.

Prob 2 (2 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, find the solution of the PDE,

$$\frac{1}{1+t} \frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = x \quad ,$$

with the boundary condition,

$$u(x, 0) = \exp(-x^2) \ .$$

In addition, find the "steady state" solution $u_s(x)$. (By definition, $u_s(x)$ is the solution $u(x,t)$ as $t \to \infty$.) Plot the solution $u(x,t)$ as a function of x at t = 0, 0.2, 0.7, and 1.5, along with the steady state $u_s(x)$. Please collect all 5 curves in one plot. It is recommended that the plot be made over the range of $-3 \leq x \leq 3$.

Prob 3 (3 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, use the method of characteristics (MOC) to solve the PDE,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad ,$$

with the boundary conditions,

$$u(x, 0) = \exp(-x^2) \ ,$$

and

$$u_t(x, 0) = 0 \ .$$

We expect a closed-form analytic solution without any unevaluated integrals. To receive full credit, you must <u>use MOC and provide the procedure</u>. Plot the solution $u(x,t)$ as a function of x at t = 0, 1, and 2. It is recommended that the plot be made over the range of $-5 \leq x \leq 5$.

Prob 4 (2 points)
For $u(x, t)$ defined on the infinite domain of $-\infty < x < \infty$ and $t \geq 0$, consider the equation,

$$\frac{\partial u}{\partial t} = \frac{u}{1+t} + Q(t) \quad ,$$

with the boundary condition,

$$u(x, 0) = \mathrm{P}(x).$$

Find the Green's function, $G(t, t')$, such that for any given $Q(t)$ and $\mathrm{P}(x)$ the solution of the system can be expressed as

$$u(x,t) = G(t,0)P(x) + \int_0^t G(t,t')Q(t')dt' \quad .$$