MAE/MSE 502, Fall 2017, HW2 Solutions

Prob 1

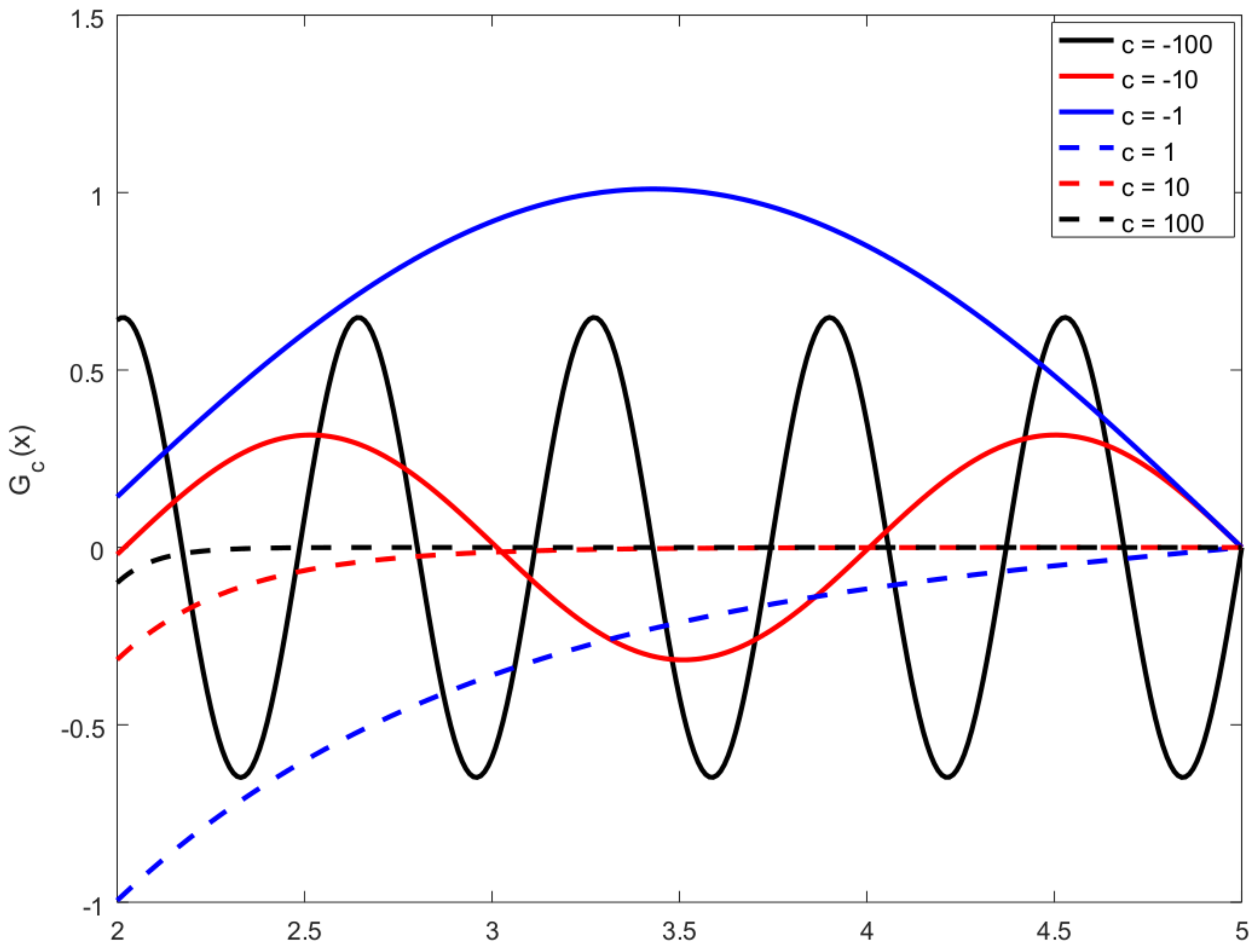
(a) The eigenvalues are continuous. All values of c (for $-\infty < c < \infty$), <u>except</u> $c = -(n\pi/6)^2$ with $n = 1, 3, 5, 7, \ldots$, are eigenvalues. The eigenfunctions are

$$G_c(x) = \frac{\sinh[(x-5)\sqrt{c}\,]}{\sqrt{c}\,\cosh(3\sqrt{c}\,)} \quad , \text{ if } c > 0$$

$$G_c(x) = x-5 \quad , \text{ if } c = 0$$

$$G_c(x) = \frac{\sin[(x-5)\sqrt{-c}\,]}{\sqrt{-c}\,\cos(3\sqrt{-c}\,)} \quad , \text{ if } c < 0 \text{ and } c \neq -(n\pi/6)^2 \text{ with } n = 1, 3, 5, 7, \ldots$$

(b) Plot:



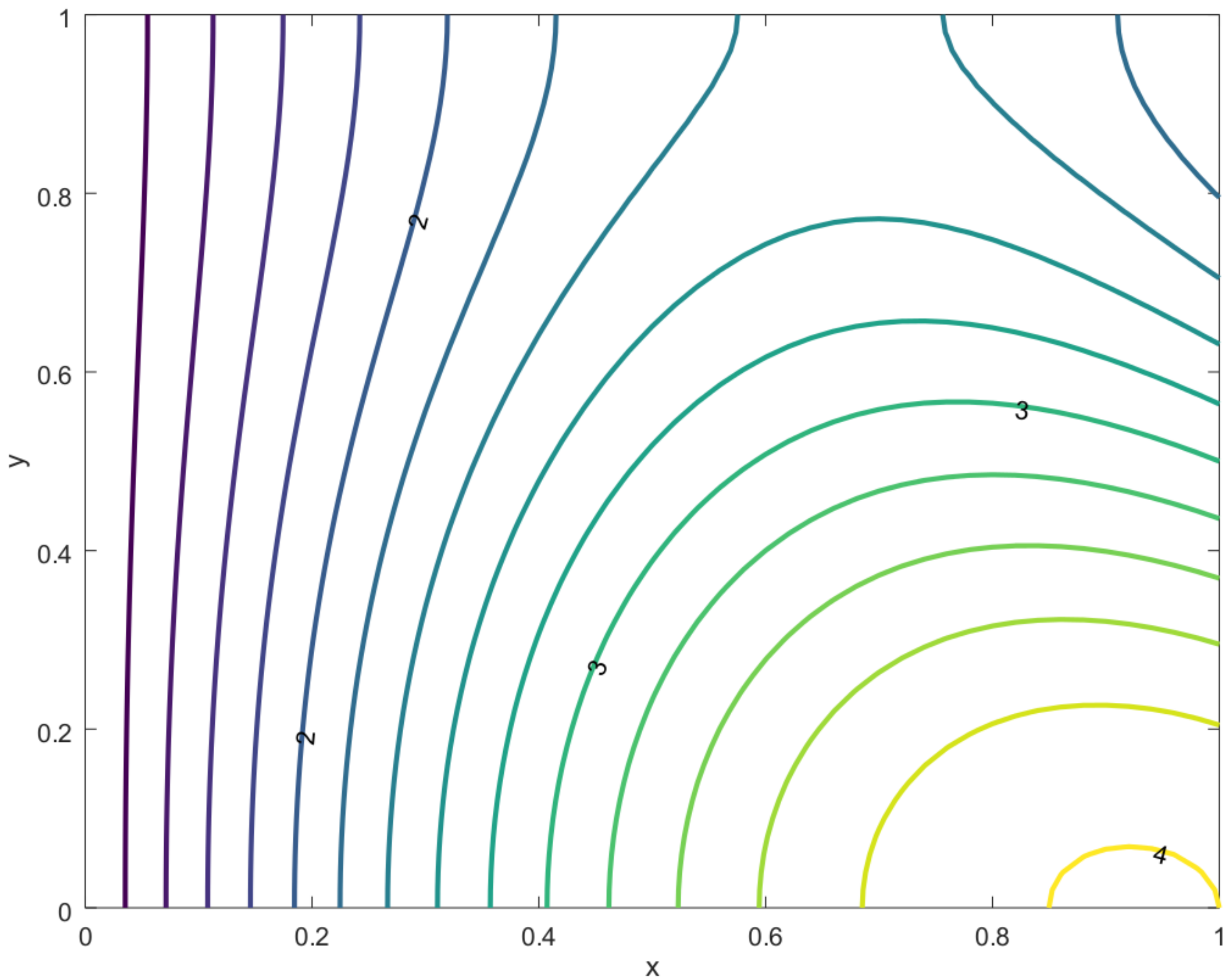
(c) The orthogonality relation does not hold. A counterexample can be readily constructed by a direct evaluation of the integral of the product of two distinctive eigenfunctions.

(d) No, since $A\,G_p(x)$ (with $A \neq 1$) does not satisfy the first boundary condition.

Prob 2

$$u(x, y) = \cos\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{\pi x}{2}\right) + x\cos(\pi y)$$

Contour plot (with contour interval = 0.2):



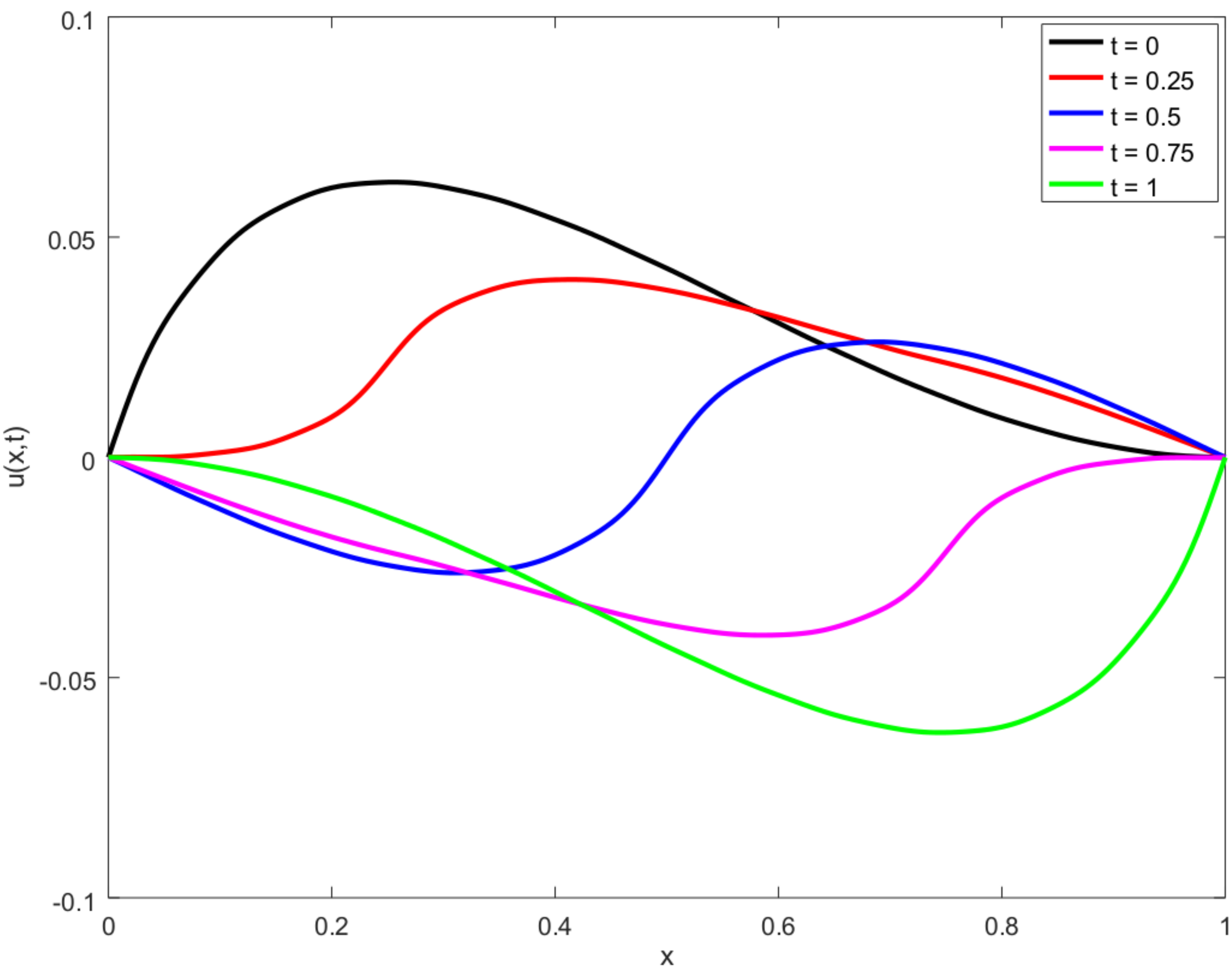
Prob 3(a)

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(n\pi t) \quad ,$$

where

$$a_n = 2\int_0^1 (\sqrt{x} - x)^2 \sin(n\pi x)\, dx \quad .$$

Plot:



Prob 3(b)

$$u(x,t) = b_1 \sin(\pi x) + \sum_{n=2}^{\infty} a_n \sin(n\pi x) \cos(\sqrt{n^2-1}\,\pi t) \quad ,$$

where

$$b_1 = 2\int_0^1 (\sqrt{x}-x)^2 \sin(\pi x)\,dx \quad \text{, and} \quad a_n = 2\int_0^1 (\sqrt{x}-x)^2 \sin(n\pi x)\,dx \quad .$$

Note: We can actually merge the first term in the solution into the summation. Nevertheless, the expression of $u(x, t)$ given above helps us see that the "first mode" associated to $\sin(\pi x)$ (which is a "positive bump") is independent of time. Oscillation in time only occurs in the higher modes with $n > 1$. This is why the plot looks like the string stays in the positive side ($u > 0$) most of the time.

Plot: