<u>Hard copy</u> of report, with a properly filled <u>cover sheet</u>, is due *at the start of class* on the due date.

Prob 1 (3 points)
Consider the eigenvalue problem for $G(x)$ defined on the interval, $2 \leq x \leq 5$,

$$\frac{d^2 G}{dx^2} = c\,G \ , \ G'(2) = 1 \ , \ G(5) = 0 \ . \quad \text{(Be aware that the 1st b.c. is on the derivative of } G.)$$

(a) Determine the eigenvalues and the corresponding eigenfunctions of this problem.
Are the eigenvalues discrete? For example, in the familiar problem when $G(x)$ is defined on
$0 \leq x \leq 1$ and the boundary conditions are $G(0) = 0$ and $G(1) = 0$, we would have $c = c_n = -n^2 \pi^2$
(n is a positive integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation
when the eigenvalues are not discrete is if all values within an interval, $A \leq c \leq B$, are valid
eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

(b) Plot the eigenfunctions, $G_C(x)$, associated to the eigenvalues $c = -100, -10, -1, 1, 10$, and 100.
(You will find in Part (a) that all those values are indeed valid eigenvalues.) Please collect all 6
curves in a single plot. Note that $G(x)$ is defined only on the interval of $2 \leq x \leq 5$. Your plot should
cover only that interval.

(c) Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_2^5 G_p(x) G_q(x) \, dx = 0, \ \text{ if } p \neq q \ , \quad \text{[Note the correction: the integral is from } x = 2 \text{ to } x = 5\text{]}$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions that correspond to two distinctive eigenvalues p
and q? Your answer should be more than just "yes" or "no". For example, in order to claim that
two eigenfunctions are not orthogonal, you may evaluate the above integral of $G_p(x) G_q(x)$ and
show that it leads to a non-zero value even when $p \neq q$. One such counterexample would suffice
to prove that the orthogonality relation does not hold. On the other hand, if you claim that the
orthogonality relation holds, you must show that it holds for all pairs of p and q.

(d) If $G_p(x)$ is an eigenfunction corresponding to an eigenvalue, $c = p$, would $A G_p(x)$ (where A is
an arbitrary constant; $A \neq 1$) also be an eigenfunction? Provide a brief explanation to support your
yes/no answer.

Prob 2 (3 points)
For $u(x, y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, solve the PDE,

$$4 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \pi^2 u = 0$$

with the boundary conditions,

(i) $u(0, y) = 1$ (ii) $u(1, y) = 3 + \cos(\pi y)$ (iii) $u_y(x, 0) = 0$ (iv) $u_y(x, 1) = 0$ (u_y is $\partial u / \partial y$)

For this problem, we expect a closed-form analytic solution with a finite number of terms and with no unevaluated integrals. A deduction will be assessed on any unevaluated integral or a sum of infinitely many terms that is left untreated in the final answer. Make a contour plot of your solution in the x-y plane. For this problem, the recommended contour interval is 0.2.
See **additional note** in next page for an example of using Matlab to make a contour plot.

Prob 3 (3 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, consider the PDE (which is a 1-D wave equation if $A = 0$),

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + A u \ ,$$

with the boundary conditions,

(i) $u(0, t) = 0$ (ii) $u(1, t) = 0$ (iii) $u(x, 0) = \left(\sqrt{x} - x \right)^2$ (iv) $u_t(x, 0) = 0$ (u_t is $\partial u / \partial t$) .

(a) Solve the system with $A = 0$. Plot the solution as a function of x at $t = 0, 0.25, 0.5, 0.75,$ and 1. Please collect all 5 curves in one plot.

(b) Solve the system with $A = \pi^2$. Plot the solution as a function of x at $t = 0, 0.25, 0.5, 0.75,$ and 1. Please collect all 5 curves in one plot. Briefly explain how the extra term in the PDE influences the solution.

Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x)\exp(-2y)$ over the domain of $0 \le x \le 1$ and $0 \le y \le 1$, using the contour levels of (-0.9, -0.7, -0.5, -0.3, -0.1, -0.05, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9). The contours for u = -0.7, -0.3, 0.3, and 0.7 are labeled. It is essential to provide the coordinates of the grid (x2d and y2d in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y. A black-and-white contour plot is acceptable as long as the contours are properly labeled.

```
clear
x = [0:0.01:1]; y = [0:0.01:1];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = sin(2*pi*x(i))*exp(-2*y(j));
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
[C,h] = contour(x2d,y2d,u,[-0.9:0.2:-0.1 -0.05 0.05 0.1:0.2:0.9]);
clabel(C,h,[-0.7 -0.3 0.3 0.7])
xlabel('x'); ylabel('y')
```