MAE/MSE 502, Fall 2017, Homework #3

Prob. 1 (3 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, consider the following three PDEs:

$$\text{(I)} \quad \frac{\partial u}{\partial t} = 4\,\frac{\partial u}{\partial x} \qquad \text{(II)} \quad \frac{\partial u}{\partial t} = 4\,\frac{\partial u}{\partial x} + 3\,\frac{\partial^2 u}{\partial x^2} \qquad \text{(III)} \quad \frac{\partial u}{\partial t} = 0.3\,\frac{\partial^3 u}{\partial x^3} \ .$$

Solve each PDE with periodic boundary conditions in the x-direction (i.e., u and all of its partial derivatives in x are periodic in the x-direction) and the boundary condition in t-direction given as

$u(x, 0) = [1 - \cos(x)]^8$. (Note the eighth power in the given function.)

You may express the solution as an infinite (Fourier) series. For each case, plot the solution as a function of x at $t = 0$, 0.1, and 0.3. Please collect all three curves in one plot. [See the general remark below HW1-Q1 on estimating the appropriate number of terms to keep in the series.]

Prob. 2 (2 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$\frac{1}{1+t}\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^5 u}{\partial x^5} + u \quad ,$$

with periodic boundary conditions in the x-direction and the boundary condition in the t-direction given as

$u(x, 0) = 1 + \cos(x) + 3\sin(x)$.

We expect a closed-form solution which consists of only a finite number of terms and no unevaluated integrals. The solution should be expressed in real numbers and functions. A deduction will be assessed on any imaginary number "i" ($= \sqrt{-1}$) that is left in the solution.

Prob 3 (2 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u + 3 + e^{-t}\cos(\pi x) \quad ,$$

with the boundary conditions

(i) $u_x(0, t) = 0$ (ii) $u_x(1, t) = 0$ (iii) $u(x, 0) = 3 + \cos(\pi x) + \cos(2\pi x)$.

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals.

Prob 4 (2 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \cos(t) \quad ,$$

with periodic boundary conditions in the x-direction and the following boundary conditions in the t-direction,

(i) $u(x, 0) = 0$,
(ii) $u_t(x, 0) = \cos(x)$.

We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals. The solution should be expressed in real numbers and functions. A deduction will be assessed on any imaginary number "i" (= $\sqrt{-1}$) that is left in the solution.

Prob 5 (2 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, consider the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + t + \cos(\pi x) \quad ,$$

with the boundary conditions

(i) $u_x(0, t) = 1$ (ii) $u_x(1, t) = 3$ (iii) $u(x, 0) = x^2 + x$.

Since the system has a nonhomogeneous PDE and only nonhomogeneous boundary conditions, a direct application of separation of variables would not work. Also, one can readily verify that this system does not have a steady solution (since the "total energy" goes to infinity as $t \to \infty$). Therefore, the strategy of solving the "perturbation equation" for the "departure from steady state" (cf. Sec. 8.2) would not work. Nevertheless, if we consider the change of variable,

$U(x, t) \equiv u(x, t) - f(x)$,

with a chosen $f(x)$ that satisfies $f'(0) = 1$ and $f'(1) = 3$ ("prime" is derivative with respect to x), then it is guaranteed that the first two boundary conditions for U will be homogeneous. More precisely, $U_x(0, t) = 0$ and $U_x(1, t) = 0$. Writing the PDE and the 3rd boundary condition also in terms of U, we can use standard methods (*cf.* Sec 8.3) to solve $U(x, t)$. [The minor price we pay is that there will be extra term(s) in the PDE for U, but those are manageable for this problem.] Try to use this strategy to find the solution, $u(x, t)$, for the system. We expect a closed-form solution which consists of only a finite number of terms and without any unevaluated integrals.