MAE/MSE 502, Fall 2017, Homework #4

Please include the printout of your code(s) in the report. You might find the following formula useful:

$$\int_0^{\infty} e^{-x^2} \cos(2bx)\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \qquad \textbf{Eq. (1)}$$

Prob 1 (3 points)
(a) For $u(x,t)$ defined on the infinite domain of $-\infty < x < \infty$ and $t \geq 0$, use the Fourier transform method to solve the PDE (with A and B given constants),

$$\frac{\partial u}{\partial t} = A \frac{\partial u}{\partial x} + B \frac{\partial^2 u}{\partial x^2} \ ,$$

with the boundary conditions

(i) $u(x, t)$ and its partial derivatives with respect to x vanish as $x \to \pm\infty$
(ii) $u(x,0) = \exp(-x^2)$.

We expect a closed-form analytic solution without any integral in the final answer.
(b) Plot the solution $u(x, t)$ as a function of x at $t = 0$, 0.5, and 2 for the following three cases:
(i) $A = 4$, $B = 3$. (ii) $A = 4$, $B = 0$. (iii) $A = 0$, $B = 3$. In each case, collect the three curves in one plot. Make the plot over the range of $-20 \leq x \leq 20$. Briefly explain the behavior of the solution in those 3 cases.
[Note: For this problem, an analytic solution can be obtained by applying the integral formula, Eq. (1), twice. Once in the forward and once in the inverse Fourier transform.]

Prob 2 (3 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, use the method of Fourier transform to solve the PDE (with C a constant),

$$\frac{\partial u}{\partial t} = C \frac{\partial^3 u}{\partial x^3} \ ,$$

with the boundary conditions:

(i) $u(x, t)$ and its partial derivatives with respect to x vanish as $x \to \pm\infty$
(ii) $u(x,0) = \exp(-x^2)$

For this problem, it is acceptable to express the solution as an integral. With $C = 2$, plot the solution $u(x,t)$ as a function of x at $t = 0$, 0.5, and 2. Please collect all three curves in one plot. Make the plot over the range of $-20 \leq x \leq 20$.
[Note: For this problem, numerical integration (e.g., by the trapezoidal method) will be needed to evaluate $u(x, t)$ in order to make the plots. Since numerical integration cannot go all the way to ∞, one has to "truncate" the integral at a finite value of ω. This is analogous to truncating a Fourier series at a finite n. A useful way to determine where to truncate the integral is to plot, for a give t, $U(\omega, t)$ (the Fourier transform of $u(x, t)$) as a function of ω and observe how $U(\omega, t)$ decays with ω.]

Prob 3 (3 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \exp\left(-x^2 - t\right) \quad ,$$

with the boundary conditions:

(i) $u(x, t)$ and its partial derivatives with respect to x vanish as $x \to \pm\infty$
(ii) $u(x,0) = 0$

For this problem, it is acceptable to express the solution as an integral. Plot the solution $u(x,t)$ as a function of x at $t = 0$, 0.2, 0.5, 1, and 5. Please collect all five curves in one plot. Make the plot over the range of $-10 \leq x \leq 10$.
[Note: The plot is an important part of the answer. Expect at least 50% deduction without the plot. Please clearly label the curves in the plot. Expect a deduction if labeling is unclear, incorrect, or missing. Like Problem 2, numerical integration (e.g., by the trapezoidal method) is needed to evaluate $u(x, t)$ in order to make the plot. See remarks below Problem 2 about how to truncate the Fourier integral.]

Prob 4 (2 points)
For $u(x, y, t)$ defined on the domain of $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $t \geq 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = 4\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u \quad ,$$

with the boundary conditions

(i) $u(0, y, t) = 0$
(ii) $u(1, y, t) = 0$
(iii) $u(x, 0, t) = 0$
(iv) $u(x, 1, t) = 0$
(v) $u(x, y, 0) = \sin(\pi x)\,\sin(2\pi y) + \sin(2\pi x)\,\sin(\pi y)$.

We expect a closed-form analytic solution with only a finite number of terms and without any unevaluated integral.