Hard copy of your work is due *at the start of class* on the due date. The rules for collaboration on homework will be released separately.

Prob. 1 (4 points)
For $u(x, t)$ defined on the domain of $0 \leq x \leq 1$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \cos(t) \ ,$$

with the boundary conditions (be aware that the second condition is imposed on the derivative of u),

(i) $u(0, t) = 0$, (ii) $u_x(1, t) = 0$, and (iii) $u(x, 0) = 2 x^4 - 7 x^2 + 6 x$.

Plot the solution, $u(x,t)$, as a function of x at $t = 0$, 0.01, 0.03, 0.1, and 0.3. Please collect all five curves in a single plot.

Note:
(i) We expect the solution to be expressed as an infinite series. It is part of your job to determine the appropriate number of terms to keep in the series to ensure that the solution is accurate. As a useful measure, the solution at $t = 0$ should nearly match the given initial state in the 3rd boundary condition. If they do not match, either the solution is wrong or more terms need to be retained in the series. This remark applies to all future homework problems that require the evaluation of an infinite series.
(ii) For the evaluation of the expansion coefficients in the infinite series, there is no need to carry out the integrations analytically. (If you wish to do so, please feel free to use an online integrator or similar software such as Wolfram Alpha. No need to do it by hand.) It is perfectly acceptable to evaluate the integrals numerically using Matlab or its equivalent. See additional note in the last page for an example of using Matlab to numerically evaluate integrals.

Prob. 2 (2.5 points)
For $u(x,t)$ defined on the domain of $0 \leq x \leq 1$ and $t \geq 0$, solve the PDE,

$$u + 2\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \ ,$$

with the boundary conditions,

(i) $u_x(0, t) = 0$, (ii) $u_x(1, t) = 0$, and (iii) $u(x,0) = 1 + \cos(3\pi x)$.

We expect a closed-form analytic solution with a finite number of terms. A deduction will be assessed on any unevaluated integral that is left in the final answer. Plot the solution as a function of x at $t = 0$, 0.01, 0.05, and 0.5. Please collect all four curves in one plot.

Prob. 3 (2 points)

For $u(x,t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, consider the PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{u}{1+t} ,$$

with the boundary conditions,

(i) $u_x(0, t) = 0$, (ii) $u_x(1, t) = 1$, and (iii) $u(x,0) = 0.5\, x^2$

If the "total energy" of the system is defined as

$$E(t) \equiv \int_0^1 u(x,t)\, dx ,$$

find the exact (analytic) expression of $E(t)$. Evaluate $E(t)$ at $t = 1$. What is the value of $E(t)$ as $t \to \infty$? Based on this analysis, does a steady state exist for this system? (You are not required to find the actual steady state, if it exists. You may or may not need to find the full solution, $u(x, t)$, for all x and t, in order to answer the key questions.)

Prob. 4 (0.5 point)

For the heat transfer problem, the Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K \frac{\partial^2 u}{\partial \hat{x}^2} , \quad 0 \le \hat{x} \le L \quad (L \text{ is the length of the "metal rod", in meters}) \text{ and } \hat{t} \ge 0, \quad (1)$$

where $\hat{t}$ and $\hat{x}$ are time in seconds and distance in meters, K is thermal diffusivity in m^2/s, and u is temperature. In our class, we usually consider the non-dimensionalized version of (1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} , \quad 0 \le x \le 1 \text{ and } t \ge 0 , \quad\quad\quad (2)$$

where x is related to $\hat{x}$ by $\hat{x} = L\,x$. In (2), time is also non-dimensionalized by $\hat{t} = T\,t$, where T is a certain dimensional time scale, and so on.

(a) In order to claim that the non-dimensionalized system, (1), is equivalent to its dimensional counterpart, (2), the three parameters K, L, and T must satisfy a unique relation. First, find out what this relation is. (For the discussion in part (b) & (c), it is useful to write the relation as $T = f(K, L)$.)

(b) Suppose that the non-dimensionalized Heat equation, (2), is used to model the real world problem of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem?

(c) Same as (b), but suppose that (2) describes heat transfer along a wooden stick that is 0.3 meter long and made of pine wood ($K \approx 10^{-7}$ m^2/s), what would be the actual time, in seconds, that $t = 0.01$ corresponds to? (We consider $t = 0.01$ because it is about the time when a significant redistribution of temperature begins to take place in the scenario described in Part (d).)

(d) Are the time scales you obtained in (b) and (c) consistent with daily experience? For instance, one can use a long wooden spoon to continuously stir a boiling pot of soup without getting one's hand burned. In contrast, the same practice would make one very uncomfortable if the spoon is made entirely of copper. Note that the time scale for cooking a pot of soup is about 10 minutes. The length of a big wooden spoon is about a foot, or 0.3 meter.

Additional note: How to use Matlab to numerically evaluate an integral

One of the simplest Matlab functions for this purpose is **trapz**. It uses the trapezoidal method to evaluate an integral. For example, to numerically evaluate

$$I = \int_0^1 \sin(x)\, dx$$

with $\Delta x = 0.01$, we first construct the discretized arrays of the coordinate points and the values of the integrand at those points. We then call **trapz** with those two arrays as the input to complete the integration. The Matlab code is very simple:

```
x = [0:0.01:1];
y = sin(x);
Integ = trapz(x,y)
```

One can readily verify the outcome with the analytic result of $I = 1 - \cos(1)$.