MAE/MSE502, Spring 2017 Homework #2

Hard copy of report, with a properly filled cover sheet, is due *at the start of class* on the due date.

Prob 1 (2 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 5$ and $t \ge 0$, consider the 1-D Wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad ,$$

with the boundary conditions,

(i) $u(0, t) = 0$ (ii) $u(5, t) = 0$ (iii) $u(x, 0) = \mathrm{P}(x)$ (iv) $u_t(x, 0) = 0$ (u_t is $\partial u/\partial t$) ,

(a) Solve the system with P(x) given as

$$\begin{aligned} \mathrm{P}(x) &= -x/4 \qquad &, \text{if } 0 \le x \le 4 \\ &= (x - 5) \qquad &, \text{if } 4 < x \le 5 \,, \end{aligned}$$

and plot the solution as a function of x at t = 0, 1.5, 2.5, 3.5, 5, and 9. Please collect all 6 curves in one plot.

(b) Solve the system with P(x) given as

$$\mathrm{P}(x) = 0.2 \sin(15.6\,\pi\, x) + 0.6 \sin(15.8\,\pi\, x) + \sin(16\,\pi\, x) + 0.6 \sin(16.2\,\pi\, x) + 0.2 \sin(16.4\,\pi\, x) \,,$$

and plot the solution as a function of x at t = 0, 1.25, 2.5, and 5. For this part only, it is recommended that the four curves be plotted separately (for example, in four panels using the "subplot" function in Matlab).

Prob 2 (3 points)
For $u(x, y)$ defined on the domain of $0 \le x \le 1$ and $0 \le y \le 1$, solve the PDE,

$$4\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0$$

with the boundary conditions,

(i) $u_x(0, y) = 0$ (ii) $u_x(1, y) = 0$ (iii) $u(x, 0) = 2$ (iv) $u(x, 1) = 1 + \cos(\pi\, x)$

For this problem, we expect a closed-form analytic solution with a finite number of terms and with no unevaluated integrals. A deduction will be assessed on any unevaluated integral or a sum of infinitely many terms that is left untreated in the final answer. Make a contour plot of your solution in the x-y plane. See **Additional Note** for an example of Matlab code for making a contour plot. For this problem, the recommended contour interval is 0.1.

Prob 3 (3 points)
For $u(x, y, t)$ defined on the triangular domain in the x-y plane as shown (shaded area) in Fig. 1 and with $t \geq 0$, consider the PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 3u \quad ,$$

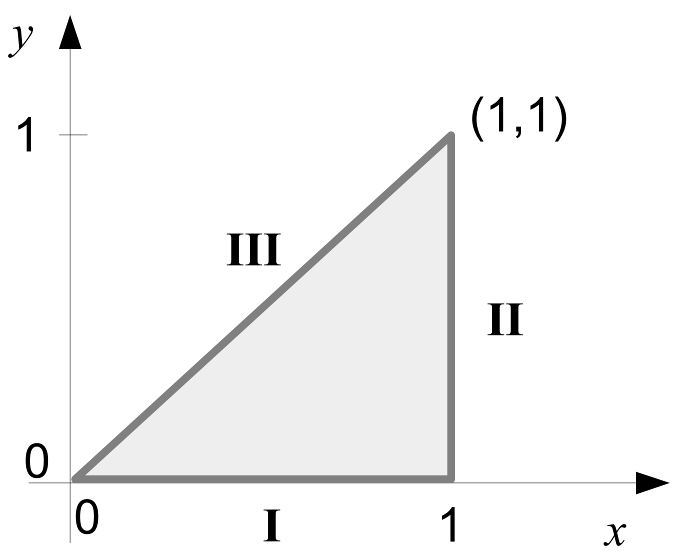
with the boundary conditions,

(i) $u_y(x, 0, t) = 0$, along the boundary segment **I**
(ii) $u_x(1, y, t) = 3$, along segment **II**
(iii) $u_x(x, y, t) = u_y(x, y, t) + 1$, along segment **III** (i.e., the line with $x = y$)
(iv) $u(x, y, 0) = x^2 + y^2 + x$

If the "total energy" of the system is defined as

$$E(t) \equiv \iint_A u(x, y, t)\, dx\, dy \quad ,$$

where the integration is over the triangular domain, find the analytic expression of $E(t)$. What is the value of $E(t)$ as $t \to \infty$?



Prob 4 (2 points)
Consider the eigenvalue problem for $G(x)$ defined on the interval, $0 \le x \le 1$,

$$\frac{d^2 G}{dx^2} = c\ G \quad , \quad G(0) = 1 \ , \ G'(1) = 2 \ .$$ (Be aware that the 2nd b.c. is on the derivative of G.)

(a) Determine the eigenvalues and the corresponding eigenfunctions of this problem. Are the eigenvalues discrete? For example, if the boundary conditions are replaced by the familiar $G(0) = 0$ and $G(1) = 0$, we would have $c = c_n = -n^2 \pi^2$ (n is an integer) as the eigenvalues. In that case, the eigenvalues are discrete. A situation when the eigenvalues are not discrete is if all values within an interval, $A \le c \le B$, are valid eigenvalues. We call the interval a *continuum*, which contains *continuous eigenvalues*.

(b) Plot the eigenfunctions, $G_C(x)$, associated with the eigenvalues c = –600, –30, –1, 0, 1, 30, and 600. (You will find in Part (a) that all those values are indeed valid eigenvalues.) Please collect all 7 curves in a single plot. Note that $G(x)$ is defined only on the interval of $0 \le x \le 1$. Your plot should cover only that interval.

(c) Do the eigenfunctions of this problem satisfy the orthogonality relation,

$$\int_0^1 G_p(x) G_q(x)\, dx = 0, \quad if\ p \ne q \quad ,$$

where $G_p(x)$ and $G_q(x)$ are two eigenfunctions that correspond to two distinctive eigenvalues p and q? Your answer should be more than just "yes" or "no". For example, in order to claim that two eigenfunctions are not orthogonal, you may evaluate the above integral of $G_p(x)G_q(x)$ and show that it leads to a non-zero value even when $p \ne q$. One such counterexample would suffice to prove that the orthogonality relation does not hold. On the other hand, if you claim that the orthogonality relation holds, you must show that it holds for all pairs of p and q.

(d) If $G_p(x)$ is an eigenfunction corresponding to an eigenvalue, $c = p$, would $AG_p(x)$ (where A is an arbitrary constant; $A \ne 1$) also be an eigenfunction? Provide a brief explanation to support your yes/no answer.

Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x)\exp(-2y)$ over the domain of $0 \le x \le 1$ and $0 \le y \le 1$, using the contour levels of (-0.9, -0.7, -0.5, -0.3, -0.1, -0.05, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9). The contours for u = -0.7, -0.3, 0.3, and 0.7 are labeled. It is essential to provide the coordinates of the grid (x2d and y2d in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y. A black-and-white contour plot is acceptable as long as the contours are properly labeled.

```
clear
x = [0:0.01:1]; y = [0:0.01:1];
for i = 1:length(x)
    for j = 1:length(y)
        u(i,j) = sin(2*pi*x(i))*exp(-2*y(j));
        x2d(i,j) = x(i);
        y2d(i,j) = y(j);
    end
end
[C,h] = contour(x2d,y2d,u,[-0.9:0.2:-0.1 -0.05 0.05 0.1:0.2:0.9]);
clabel(C,h,[-0.7 -0.3 0.3 0.7])
xlabel('x'); ylabel('y')
```